Unit 2: Working with Integers

This unit emphasizes the rules related to adding, subtracting, multiplying, and dividing integers.

Unit Focus

Number Sense, Concepts, and Operations

- Associate verbal names, written word names, and standard numerals with integers, rational numbers, and real numbers. (MA.A.1.4.1)
- Understand the relative size of integers, rational numbers, and real numbers. (MA.A.1.4.2)
- Understand concrete and symbolic representations of real numbers in real-world situations. (MA.A.1.4.3)
- Understand and use the real number system. (MA.A.2.4.2)
- Understand and explain the effects of addition, subtraction, multiplication, and division on real numbers, including square roots, exponents, and appropriate inverse relationships. (MA.A.3.4.1)
- Add, subtract, multiply, and divide real numbers, including square roots and exponents, using appropriate methods of computing, such as mental mathematics, paper and pencil, and calculator. (MA.A.3.4.3)
- Use estimation strategies in complex situations to predict results and to check the reasonableness of results. (MA.A.4.4.1)

Measurement

- Solve real-world problems involving rated measures (miles per hour, feet per second). (MA.B.2.4.2)
- Solve real-world and mathematical problems involving estimates of measurements, including length, time, weight/mass, temperature, money, perimeter, area, and volume and estimate the effects of measurement errors on calculations. (MA.B.3.4.1)

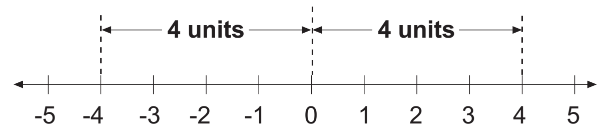
Algebraic Thinking

- Represent real-world problem situations using finite graphs. (MA.D.2.4.1)
- Use equations and inequalities to solve real-world problems graphically and algebraically. (MA.D.2.4.2)

Vocabulary

Use the vocabulary words and definitions below as a reference for this unit.

absolute value .... a number's distance from zero (0) on a number line; distance expressed as a positive value
Example: The absolute value of both 4, written $|4|$, and negative 4, written $|-4|$, equals 4.



addition property of equality ..... adding the same number to each side of an equation results in an equivalent equation; for any real numbers *a*, *b*, and *c*, if $a = b$, then $a + c = b + c$

additive inverse property .... a number and its additive inverse have a sum of zero (0)
Example: In the equation $3 + -3 = 0$, 3 and -3 are additive inverses of each other.

additive inverses .... a number and its opposite whose sum is zero (0); also called *opposites*
Example: In the equation $3 + -3 = 0$, 3 and -3 are additive inverses, or *opposites*, of each other.

coordinate .... the number paired with a point on the number line

decrease .... to make less

difference .... a number that is the result of subtraction
Example: In $16 - 9 = 7$,
7 is the difference.

factor .... a number or expression that divides evenly into another number
Example: 1, 2, 4, 5, 10, and 20 are factors of 20 and $(x + 1)$ is one of the factors of $(x^2 - 1)$.

graph of a number .... the point on a number line paired with the number

increase .... to make greater

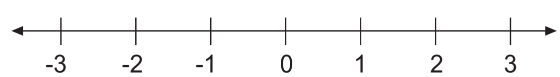
integers .... the numbers in the set
{… , -4, -3, -2, -1, 0, 1, 2, 3, 4, …}

multiplicative inverse
(reciprocal) .... any two numbers with a product of 1
Example: 4 and $\frac{1}{4}$; zero (0) has no multiplicative inverse

natural numbers
(counting numbers) .... the numbers in the set
{1, 2, 3, 4, 5, …}

negative numbers .... numbers less than zero

number line .... a line on which ordered numbers can be written or visualized



opposites .... two numbers whose sum is zero
Example: $-5 + 5 = 0$ *or* $\frac{2}{3} + \left(-\frac{2}{3}\right) = 0$
opposites opposites

origin .... the point of intersection of the x- and y-axes in a rectangular coordinate system, where the x-coordinate and y-coordinate are both zero (0)

point .... a specific location in space that has no discernable length or width

positive numbers .... numbers greater than zero

product .... the result of multiplying numbers together
Example: In 6 x 8 = 48,
48 is the product.

quotient .... the result of dividing two numbers
Example: In 42 ÷ 7 = 6,
6 is the quotient.

solve .... to find all numbers that make an equation or inequality true

subtraction property of equality .... subtracting the same number from each side of an equation results in an equivalent equation; for any real numbers a, b, and c, if a = b, then
$a - c = b - c$

sum .... the result of adding numbers together
Example: In 6 + 8 = 14,
14 is the sum.

whole number .... the numbers in the set {0, 1, 2, 3, 4, …}

Unit 2: Working with Integers

Introduction

As a very young child, you learned to count and probably started with 1, counting 1, 2, 3, and so on. The counting numbers are known as the set of *natural numbers*. You later added the number zero (0) to your set of numbers although you did not include it when counting. Our set of *whole numbers* {0, 1, 2, 3, 4, …} is important when considering the set of *integers* {… , -4, -3, -2, 0, 1, 2, 3, 4, … }.

Your work with *whole numbers* {0, 1, 2, 3, 4, …} has been extensive in the past. Your work with *integers* has not been as extensive because the need for them in the real world is less frequent. We will review the four operations of addition, subtraction, multiplication, and division with integers since these operations represent necessary tools for solving equations algebraically.

Lesson One Purpose

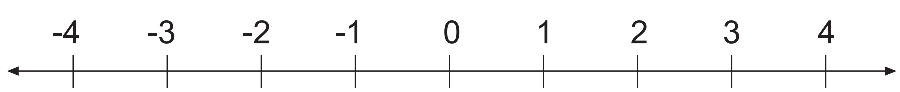
- Associate verbal names, written word names, and standard numerals with integers, rational numbers, and real numbers. (MA.A.1.4.1)
- Understand the relative size of integers, rational numbers, and real numbers. (MA.A.1.4.2)
- Understand concrete and symbolic representations of real numbers in real-world situations. (MA.A.1.4.3)
- Understand and use the real number system. (MA.A.2.4.2)
- Understand and explain the effects of addition, subtraction, multiplication, and division on real numbers, including square roots, exponents, and appropriate inverse relationships. (MA.A.3.4.1)
- Add, subtract, multiply, and divide real numbers, including square roots and exponents, using appropriate methods of computing, such as mental mathematics, paper and pencil, and calculator. (MA.A.3.4.3)

- Use estimation strategies in complex situations to predict results and to check the reasonableness of results. (MA.A.4.4.1)
- Solve real-world problems involving rated measures (miles per hour, feet per second). (MA.B.2.4.2)
- Solve real-world and mathematical problems involving estimates of measurements, including length, time, weight/mass, temperature, money, perimeter, area, and volume and estimate the effects of measurement errors on calculations. (MA.B.3.4.1)
- Represent real-world problem situations using finite graphs. (MA.D.2.4.1)
- Use equations and inequalities to solve real-world problems graphically and algebraically. (MA.D.2.4.2)

Whole Numbers, Natural Numbers, and Integers

Numbers were invented by people. The **positive numbers** 1, 2, 3, 4, 5, ... were probably invented first, and were used for counting. *Counting numbers* are also known as **natural numbers** {1, 2, 3, 4, 5, ... }. *Positive numbers* are numbers greater than zero. **Negative numbers** were invented to represent things like temperatures below freezing, overdrawn bank balances, owing money, loss, or going backwards. *Negative numbers* are numbers less than zero.

We frequently represent numbers on a **number line**.



number line

Numbers to the *left* of zero are *negative*, while numbers to the *right* are *positive*. Zero is neither positive nor negative. The set of numbers used on the number line above is called the *set of integers*.

It is helpful to define **integers** {… , -4, -3, -2, -1, 0, 1, 2, 3, 4, …} as the set of **whole numbers** {0, 1, 2, 3, 4, …} and their **opposites**. The opposite of 1 is -1 and the opposite of 2 is -2. The opposite of -5 is 5 and the opposite of -7 is 7.

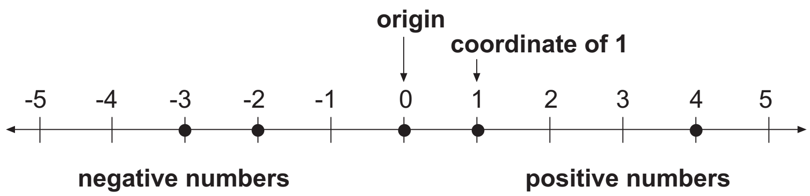
This *set* can be written {… , -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, …}.

The larger the number, the farther it is to the right. The smaller the number, the farther it is to the left. A number is considered positive even if it does *not* have a *positive sign* ("+") written in front of it.

Graph of a Number

To **graph a number** means to draw a dot at the **point** that represents that *integer*. A *point* represents an exact location. The number paired with a point is called the **coordinate** of the point. The graph of zero (0) on a number line is called the **origin**.

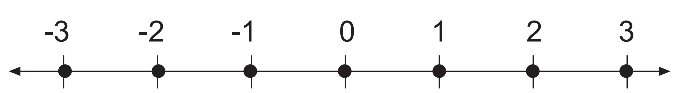
Below is a graph of {-3, -2, 0, 1, 4}.



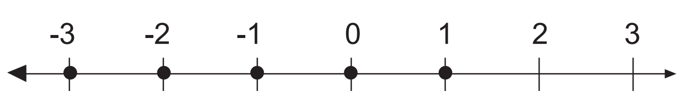
Number line above shows coordinates of points -3, -2, 0, 1, and 4.

Consider the following sets of numbers graphed below:

Graph A: The set of integers



Graph B: The set of integers less than 2



Remember: The numbers between 0 and 1 such as $\frac{1}{2}$ and 0.3 are not integers. We therefore shade the points to be included but not the line itself. Notice the arrow on the left of Graph B indicates that all the integers to the left are included in the set.

Addition of Integers

Looking for Patterns

Patterns are predictable. Study the problems below for patterns.

3 + 4 = 7	10 + 6 = 16	-5 + 8 = 3
3 + 3 = 6	10 + 4 = 14	-5 + 7 = 2
3 + 2 = 5	10 + 2 = 12	-5 + 6 = 1
3 + 1 = 4	10 + 0 = 10	-5 + 5 = 0
3 + 0 = 3	10 + -2 = 8	-5 + 4 = -1
3 + -1 = 2	10 + -4 = 6	-5 + 3 = -2
3 + -2 = 1	10 + -6 = 4	-5 + 2 = -3
3 + -3 = 0	10 + -8 = 2	-5 + 1 = -4
3 + -4 = -1	10 + -10 = 0	-5 + 0 = -5
3 + -5 = -2	10 + -12 = -2	-5 + -1 = -6

Pattern 1

$3 + -3 = 0$

and

$10 + -10 = 0$

and

$-5 + 5 = 0$

From the pattern above, we can say the following:

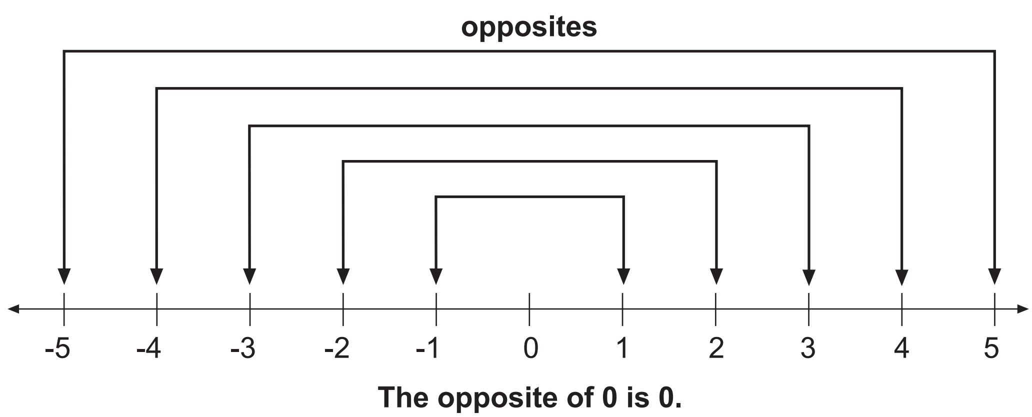
- The **sum** of a number and its opposite is zero.
- A number and its opposite are called **additive inverses** of each other.
- The **additive inverse property** tells us the following.

  For every number *n*,

  $n + (-n) = 0.$

Review of Opposites or Additives Inverses

5 and -5 are called *opposites*. Opposites are two numbers whose points on the number line are the same distance from 0 but in opposite directions.



opposites or additive inverses

Every positive integer can be paired with a negative integer. These pairs are called *opposites*. For example, the opposite of 4 is -4 and the opposite of -5 is 5.

The opposite of 4 can be written -(4) , so -(4) equals -4.

$$-(4) = -4$$

The opposite of -5 can be written -(-5), so -(-5) equals 5.

$$-(-5) = 5$$

Two numbers are opposites or *additive inverses* of each other if their sum is zero.

For example: $4 + -4 = 0$
$-5 + 5 = 0$

Additive Inverse Property

A number and its additive inverse have a sum of zero (0).

$$3 + -3 = 0$$

3 and -3 are additive inverses, or *opposites*, of each other.

Pattern 2

3 + 5 = 8

and

-5 + -1 = -6

From these and many other examples, we could say the following:

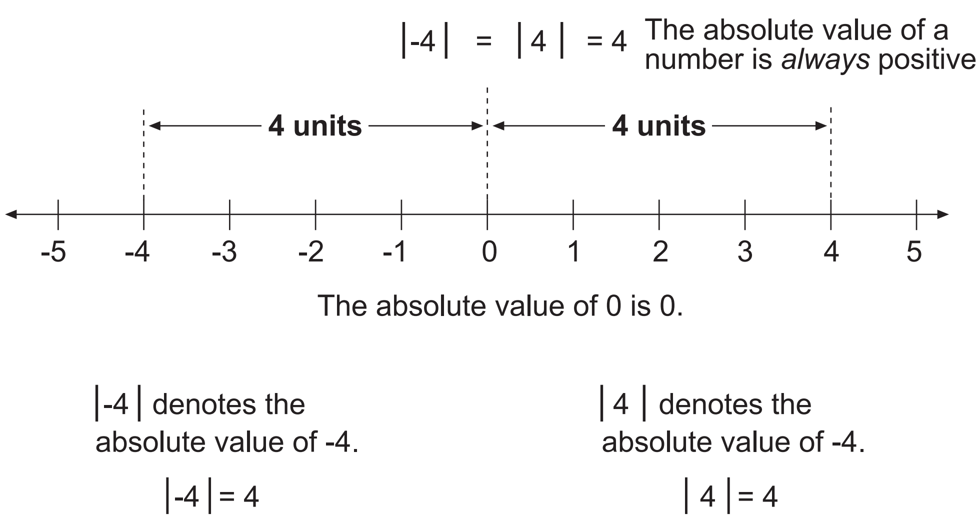
- The *sum* of two positive integers, or the sum of two negative integers, seems to be the *sum of the two amounts* and
- the *sign is the same as the sign of the two integers*.

  Another way to think of this is the sum of each number's distance from zero or the sum of their **absolute values**, with the sign the same as the two integers.

Review of Absolute Value

The *absolute value* of a number is the distance the number is from the *origin* or zero (0) on a number line. The symbol | | placed on either side of a number is used to show absolute value.

Look at the number line below. You see that -4 and 4 are different numbers. However, they are the same distance in number of units from 0. Both have the *absolute value* of 4.



The absolute value of 10 is 10. We can use this notation:

$$|10| = 10$$

The absolute value of -10 is also 10. We can use this notation:

$$|-10| = 10$$

Both 10 and -10 are 10 units away from the origin. Consequently, the absolute value of both numbers is 10.

Now that we have reviewed absolute value, we can introduce two rules for adding numbers which will enable us to add quickly.

Pattern 3

3 + -2 = 1

and

10 + -8 = 2

and

-5 + 6 = 1

From the pattern above, we can say the following:

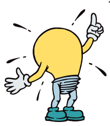
- The sum of a positive integer and a negative integer seems to be the numerical answer in their difference, and
- the sign is the same as the sign of the number the greatest distance from zero.

Review of Adding Positive and Negative Integers

There are specific rules for adding positive and negative numbers.

1. **If the two integers have the *same sign*, *keep* the *sign* and *add* their absolute values.**

   Example: -5 + -7

   Think: Both signs are negative.

   $|-5| = 5$

   $|-7| = 7$

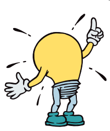
   $5 + 7 = 12$

   The sign will be negative because both signs were negative. Therefore, the answer is -12.

   $-5 + -7 = -12$

2. **If the two integers have *opposite signs*, *subtract* the absolute values. The answer has the *sign* of the integer with the *greater* absolute value.**

   Example: -8 + 3

   Think: Signs are opposite.

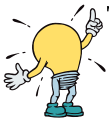
   $|-8| = 8$

   $|\ 3| = 3$

   $8 - 3 = 5$

   The sign will be negative because 8 has the greater absolute value. Therefore, the answer is -5.

   $-8 + 3 = -5$



Example: -6 + 8

Think: Signs are opposite.

$|-6| = 6$

$|\ 8| = 8$

$8 - 6 = 2$

The sign will be positive because 8 has a greater absolute value. Therefore, the answer is 2.

$-6 + 8 = 2$

Rules to Add Integers

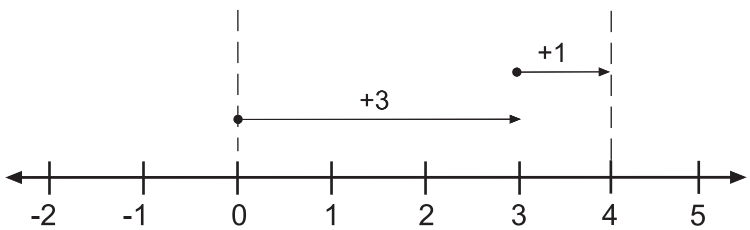
- The sum of two positive integers is *positive*.
- The sum of two negative integers is *negative*.
- The sum of a positive integer and a negative integer takes the *sign of the number with the greater absolute value*.
- The sum of a positive integer and a negative integer is zero if numbers have the *same absolute value*.

Note: The *sum* of *a positive integer and a negative integer* may be *positive, negative,* or *zero*.

Using the Number Line Method to Model Adding Positive and Negative Integers

Example 1:

3 + 1 = 4



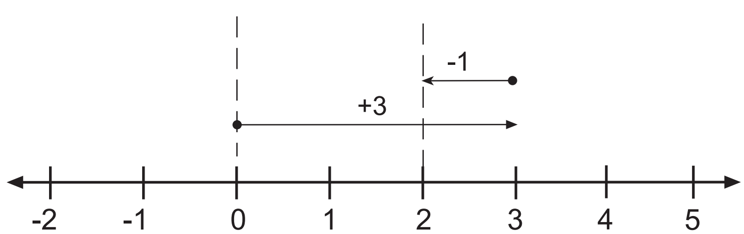
3 + 1 = 4

Think and Visualize

- From 0, move 3 units to the *right* to represent positive 3.
- From 3, move 1 unit to the *right* to add 1.
- The sum is 4.

Example 2:

3 + -1 = 2



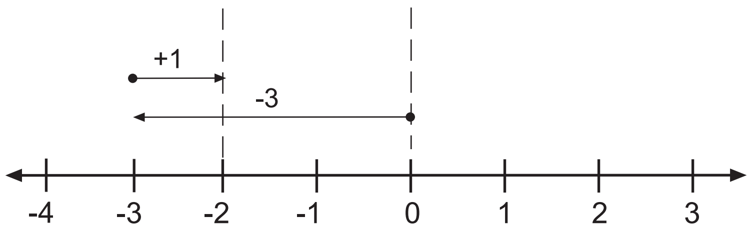
3 + -1 = 2

Think and Visualize

- From 0, move 3 units to the *right* to represent positive 3.
- From 3, move 1 unit to the *left* to add -1.
- The sum is 2.

Example 3:

-3 + 1 = -2



-3 + 1 = -2

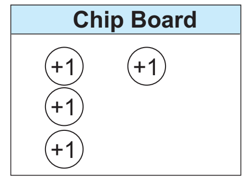
Think and Visualize

- From 0 move 3 units to the *left* to represent negative 3.
- From negative 3, move 1 unit to the *right* to add 1.
- The sum is -2.

Using the Chip Method to Model Adding Positive and Negative Integers

Let the unshaded chips represent positive numbers and the shaded chips represent negative numbers.

Example 1:

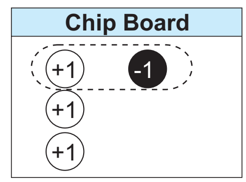


$3 + 1 = 4$

Think and Visualize

- The sum of 4 unshaded chips, each with a value of +1, is 4.

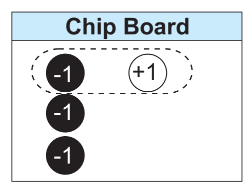
Example 2:



$3 + -1 = 2$

Think and Visualize

- The sum of +1 and -1 is zero. These two chips can be removed from the chip board.
- 2 chips, each with a value of +1, remain.
- The sum is 2.



Example 3:

$-3 + 1 = -2$

Think and Visualize

- The sum of +1 and -1 is zero. These two chips can be removed from the chip board.
- 2 chips, each with a value of -1, remain.
- The sum is -2.

Practice

Use the **number line method** *or the* **chip method** *to solve the following.*

1. 5 + -9 = ____________

2. -8 + 10 = ____________

3. -24 + -20 = ____________

4. 50 + 76 = ____________

5. -42 + 21 = ____________

6. -7 + 9 = ____________

7. 21 + -30 = ____________

8. 1,000 + -256 = ____________

9. 84 + -92 = ____________

10. 6 + -54 = ____________

Answer the following.

11. On May 22, 2004, Annika Sorenstam became the first woman in 58 years to play on the PGA Tour. Her score qualified her to continue play on the second day. Eighteen holes were played. *Par* was 3 for four holes, 4 for twelve holes, and 5 for two holes, yielding a total par of 70 on the course in Ft. Worth, Texas. Sorenstam's scores in relation to par are provided in the table below. Find her total.

    Note: Par is the standard number of strokes a good golfer is expected to take for a certain hole on a given golf course.

Annika Sorenstam's Golf Scores for Day 2

Hole	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
Total Score Relative to Par	0	0	0	0	+1	0	0	0	+1	0	0	0	-1	0	0	0	0	0

Answer: ____________

12. On day two of the play, Sorenstam's scores did *not* qualify her to play on day three. Her scores in relation to par are provided in the table below. Find her total for day three.

Annika Sorenstam's Golf Scores for Day 3

Hole	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
Total Score Relative to Par	0	-1	0	0	+1	+1	0	+1	0	+1	0	+1	0	0	0	0	0	0

(You may be interested to know that Babe Zaharias was the last woman to compete on the PGA Tour, in 1945. She made the 36-hole cut in the Los Angeles Open but did not qualify for the final round.)

Answer: ____________

13. When an unknown integer is added to 12, the sum is *less than* -2. Give three examples of what the unknown number might be.

Answer: ____________

____

____

Complete the following statements.

14. a. The sum of *two positive numbers* is ____________ (always, sometimes, never) positive.

b. The sum of *two negative numbers* is ____________ (always, sometimes, never) positive.

c. The sum of *a number and its opposite* is ____________ (always, sometimes, never) positive.

d. The sum of *a positive number and a negative number* is ____________ (always, sometimes, never) positive.

Keeping Equations in Balance

Think about This!

We know that addition and subtraction are inverse operations since one "undoes" the other. This is often helpful in solving equations in a step-by-step method.

Consider the following problem:

Margaret earns $4 *less* an hour than Susie. If Margaret earns $20 per hour, what is Susie's hourly rate?

In the equation,

$$s - 4 = 20,$$

s represents Susie's hourly rate.

We might think:

- "From what number can I subtract 4 and the result be 20?" and solve the equation intuitively.

  $$? - 4 = 20$$

  $$24 - 4 = 20$$

  Susie's hourly rate is $24.

However, *addition*, the *inverse operation of subtraction* can also be used. Thus, to solve for s, I can eliminate the subtraction of 4 from s by adding 4. If I add 4 to the left side, I must add 4 to the right side.

$$s - 4 = 20$$

$$s - 4 + \mathbf{4} = 20 + \mathbf{4} \longleftarrow \text{add 4 to both sides} \longrightarrow \begin{array}{rcr} s - \cancel{4} & = & 20 \\ + \cancel{4} & & + 4 \\ \hline s & = & 24 \end{array}$$

$$s = 24$$

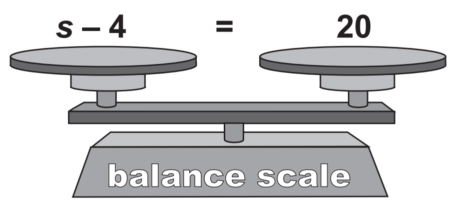
Susie's hourly rate is $24.

The value on the *left* side of the equation, $s - 4$, is the same as the value on the *right* side of the equation, 20. If I add 4 to the value on the *left* side, I must add 4 to the value on the *right* side if the equation is to remain true. This property is called the **addition property of equality**.

Equality properties help you keep an equation *balanced*. When you think of an equation, visualize it on a balance scale. See below.

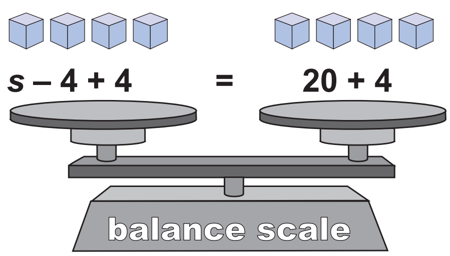
$$s - 4 = 20$$

The amount on this side balances the amount on the other side.

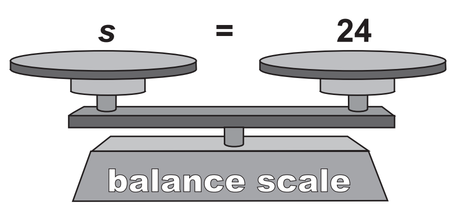


$$s - 4 + 4 = 20 + 4$$

If I add 4 to this side, I must add 4 to the other side to keep both sides balanced.



$$s = 24$$



keeping an equation balanced

Equality Properties Equality properties keep an equation in balance.	
Addition Property of Equality The same number can be added to each side of an equation and the equation remains balanced. For any real numbers a, b, and c, if $a = b$, then $a + c = b + c$.	**Subtraction Property of Equality** The same number can be subtracted from each side of an equation and the equation remains balanced. For any real numbers a, b, and c, if $a = b$, then $a - c = b - c$.

Now consider the following problem:

Jack and Tom played a game in which Tom's score was 2 points *less than* Jack's. Tom's score was -5. What was Jack's score?

In the equation, j represents Jack's score.

$$j - 2 = -5$$

$$j - 2 + \mathbf{2} = -5 + \mathbf{2} \longleftarrow \text{add 2 to both sides} \longrightarrow \begin{array}{rcr} j - \cancel{2} & = & -5 \\ + \cancel{2} & & +2 \\ \hline j & = & -3 \end{array}$$

$$j = -3$$

Jack's score was -3.

Adding or Subtracting Integers

What to Do	How to Solve	Example
• Add numbers with the same sign	• Add absolute values. • Use the original sign in answer.	-8 + (-8): |-8| + |-8| = 8 + 8 = 16 So -8 + (-8) = -16.
• Add numbers with different signs	• Subtract absolute values. • Use sign of *addend* (any number being added) with greater absolute value.	-8 + 2: |-8| – |2| = 8 – 2 = 6 |-8| > |2| So -8 + 2 = -6.
• Subtract	• Add the opposite of the number following the subtraction sign.	-8 – 5 = -8 + (-5) = -13

Practice

Use the **addition property of equality** *to solve the following. The first one has been done for you.*

1. $a - 25 = 40$

$$a - 25 + 25 = 40 + 25 \qquad a = 65$$

or

$$\begin{array}{rcr} a - \cancel{25} & = & 40 \\ \underline{+\cancel{25}} & & \underline{+\ 25} \\ a & = & 64 \end{array}$$

self-check:
Does $65 - 25 = 40$? Yes.

2. $b - 25 = 15$

3. $c - 13 = 30$

4. $d - 13 = 5$

5. $e - 12 = -2$

6. $f - 14 = -20$

7. $g - 5 = -8$

8. $h - 60 = 24$

Practice

Match each definition with the correct term. Write the letter on the line provided.

______ 1. two numbers whose sum is zero

______ 2. the numbers in the set {… , -4, -3, -2, -1, 0, 1, 2, 3, 4, …}

______ 3. the numbers in the set {1, 2, 3, 4, 5, …}

______ 4. numbers less than zero

______ 5. the numbers in the set {0, 1, 2, 3, 4, …}

______ 6. a number and its additive inverse have a sum of zero (0)

______ 7. numbers greater than zero

______ 8. adding the same number to each side of an equation results in an equivalent equation; for any real numbers a, b, and c, if $a = b$, then $a + c = b + c$

A. addition property of equality

B. additive inverse property

C. integers

D. natural numbers (counting numbers)

E. negative numbers

F. opposites

G. positive numbers

H. whole number