Lesson Two Purpose

- Understand concrete and symbolic representations of real numbers in real-world situations. (MA.A.1.4.3)
- Understand and use the real number system. (MA.A.2.4.2)
- Add, subtract, multiply, and divide real numbers, including square roots and exponents, using appropriate methods of computing, such as mental mathematics, paper and pencil, and calculator. (MA.A.3.4.3)
- Solve real-world problems involving rated measures (miles per hour, feet per second). (MA.B.2.4.2)
- Solve real-world and mathematical problems involving estimates of measurements, including length, time, weight/mass, temperature, money, perimeter, area, and volume and estimate the effects of measurement errors on calculations. (MA.B.3.4.1)
- Using a rectangular coordinate system (graph), apply and algebraically verify properties of two-dimensional figures, including distance, midpoint, slope, parallelism, and perpendicularity. (MA.C.3.4.2)
- Describe, analyze, and generalize relationships, patterns, and functions using words, symbols, variables, tables, and graphs. (MA.D.1.4.1)
- Represent real-world problem situations using finite graphs. (MA.D.2.4.1)
- Use equations and inequalities to solve real-world problems graphically and algebraically. (MA.D.2.4.2)

Equations with Two Variables

In this unit, we will study **equations** with two **variables**. We will limit our study to *equations* such as these:

$2x + y = 10$; $y = 3x + 5$; and $y = 4x$

To begin, let's see how we can **solve** the following equation.

$x + y = 5$.

You will find that, unlike the earlier equations in Unit Three, this type of equation has an **infinite** number (*no limit* to the number) of **solutions**. A *solution of an equation with two variables* is an ordered pair of numbers that make the equation true.

Suppose that we replace the variable x with the **value** of 0. Obviously, the variable y would have to be replaced with the number 5, because

$0 + 5 = 5$.

So if $x = 0$, then $y = 5$ and we use the *ordered pair* (0, 5) to denote this solution.

Here are two other solutions for $x + y = 5$.

- If we let $x = 1$, then y would be 4 because

  $1 + 4 = 5$.

Hence, (1, 4) is a solution.

- If we let $x = 2$, then y would be 3, because

  $2 + 3 = 5$.

Therefore, (2, 3) is also a solution.

There are an *infinite* number of ordered pairs that work for the equation $x + y = 5$.

Below is a **table** *of values* with several examples of ordered pairs that are solutions to the equation. A *table* is an orderly display of numerical information in rows and columns.

Table of Values

$x + y = 5$	
0	5
1	4
2	3
3	2
1	4
5	0
$\frac{1}{2}$	$4\frac{1}{2}$
0.25	4.75
⋮	⋮

Example: Find five solutions for the equation $y = x - 2$. *Values* for x were chosen and substituted in the equation. Next, y was solved for in each equation.

Choosing and Substituting Values for *x* to Solve for *y*

Equation	*x*	Substitute for *x*	Solve for *y*	Solution
$y = x - 2$	0	$y = \mathbf{0} - 2$	-2	(0, -2)
	1	$y = \mathbf{1} - 2$	-1	(1, -1)
	2	$y = \mathbf{2} - 2$	0	(2, 0)
	-1	$y = \mathbf{-1} - 2$	-3	(-1, -3)
	-5	$y = \mathbf{-5} - 2$	-7	(-5, -7)

Remember: In an ordered pair, the value of x is listed first, and the value for y is listed second—even if the variable y is written first in the equation.

Example: In the equation $y = x - 2$, the solution is (0, -2).

$$-2 = 0 - 2$$
$$-2 = -2$$

$x \quad y$

Graphing Linear Equations

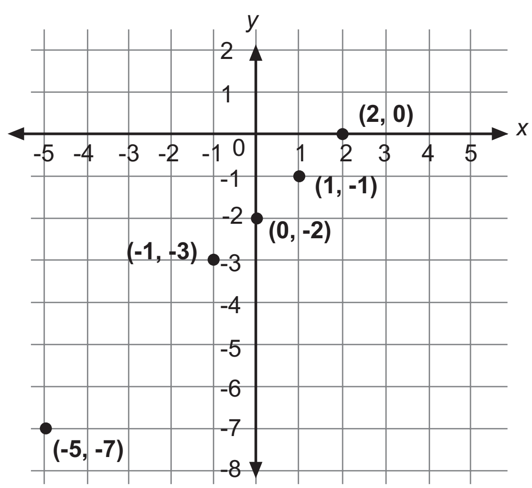
Referring to the previous section, we found five solutions for the equation $y = x - 2$. Below is a *table of values* with the summary of the results.

Table of Values

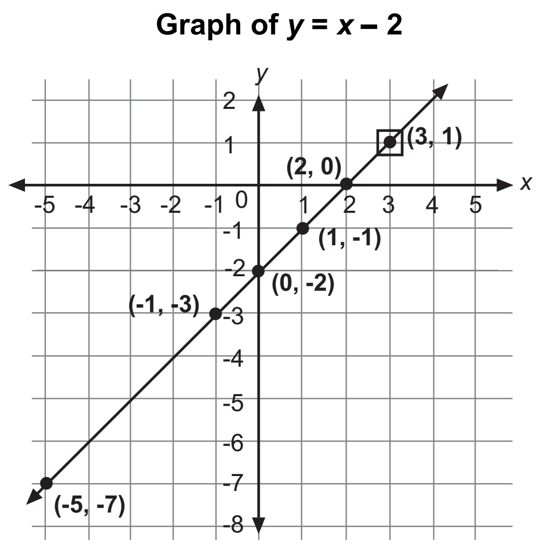
$y = x - 2$		
x	y	
0	-2	(0, -2)
1	-1	(1, -1)
2	0	(2, 0)
-1	-3	(-1, -3)
-5	-7	(-5, -7)

The **graph of an equation** with two variables is all the points whose coordinates are solutions of the equation. The *coordinates* correspond to points on a *graph*. Let's plot these *points* on the graph and see what we get.

Graph of $y = x - 2$



Carefully draw the line that connects these points.



The graph of the equation $y = x - 2$ is the line drawn on the coordinate plane. The coordinate grid itself contains the *x*-axis—the horizontal (⟷) axis and *y*-axis—the vertical (↕) axis. The *line* drawn is endless in **length (*l*)** and the *plane* is a flat surface with no boundaries.

Notice that the point (3, 1) lies on our line, but it was *not* one of our original points. Let's see if it is a solution. **Substitute** or *replace* the variables *x* and *y* with (3, 1).

$y = x - 2$ ← substitute 3 for *x* and 1 for *y*

$1 = 3 - 2$

$1 = 1$ This is true, so (3, 1) is a solution of the equation.

It turns out that the coordinates of any point on the line is a solution of the equation. We cannot write all the solutions to an equation because *x* can be anything, but we can draw a *picture* of the solutions using the coordinate plane and a line. The equations we have been working with in the last section are called **linear equations** because their graphs are always *straight lines*.

Practice

Match each definition with the correct term. Write the letter on the line provided.

______	1. all points whose coordinates are solutions of an equation	A. equation
______	2. a mathematical sentence in which two expressions are connected by an equality symbol	B. graph of an equation
______	3. any symbol, usually a letter, which could represent a number	C. infinite
______	4. having no boundaries or limits	D. linear equation
______	5. an algebraic equation in which the variable quantity or quantities are raised to the zero or first power and the graph is a straight line	E. solution
______	6. any value for a variable that makes an equation or inequality a true statement	F. solve
______	7. to replace a variable with a numeral	G. substitute
______	8. to find all numbers that make an equation or inequality true	H. value (of a variable)
______	9. any of the numbers represented by the variable	I. variable

Linear Graphs

For each of the following you will consider an *equation*, a *table*, and a *graph*.

Example: The **formula** for **perimeter** of a **square** is $P = 4s$ where P represents *perimeter* and s represents the *length (l)* of a *side*.

- **Equation:**

  $P = 4s$ where P represents perimeter and s represents the length of a side.

This establishes a **pattern** or **relationship** between the length of a side and the perimeter of a *square*. The perimeter is 4 times the length of a side.

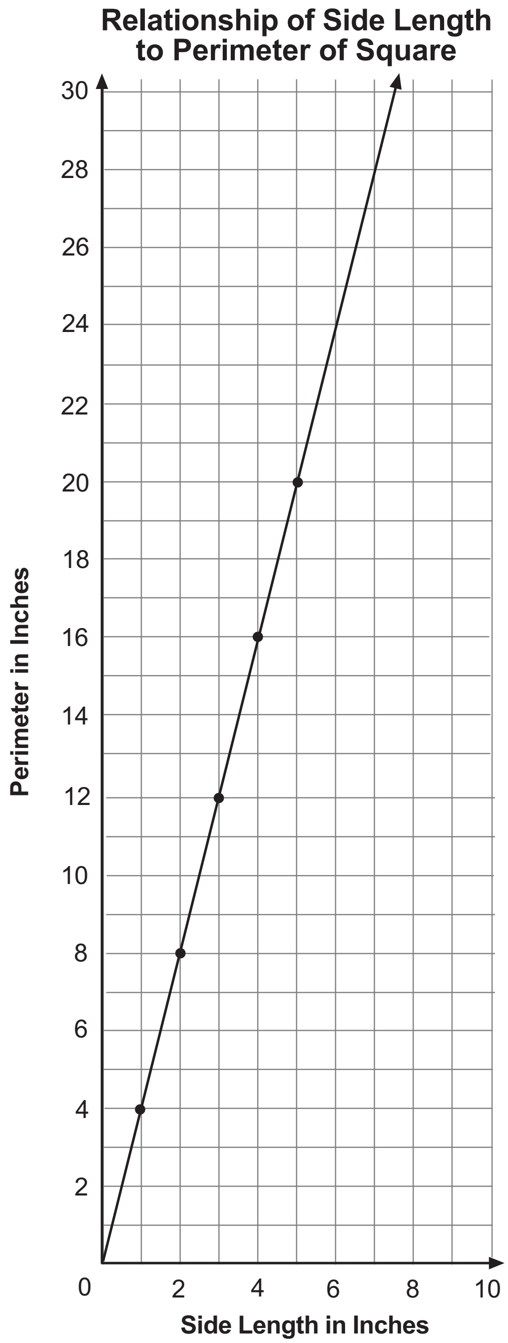
- **Table:**

Relationship of Side Length to Perimeter of a Square

Length of Side s	Perimeter of Square $P = 4s$
1 inch	4 inches
2 inches	8 inches
3 inches	12 inches
4 inches	16 inches
5 inches	20 inches
s inches	$4s$ inches

Notice that as the length of a side increases by 1, the perimeter increases by 4. This represents a constant **rate of change**.

- **Graph:**



When each pair of coordinates from the table is plotted on the coordinate grid, we observe they are *linear*, which means they form a straight line. Since the side lengths of a square could be any positive number, the points can be connected.

Summary for this problem:

The equation was $P = 4s$, where P represented the perimeter of a square and s represented the length of a side.

- The equation indicates a *relationship* between the *perimeter of a square* and its *side length*.
- As the values for side length in the table increased by 1, the values for perimeter increased by 4. This represents a *constant rate of change*.
- When each pair of coordinates was plotted on a coordinate grid, the result was linear. They formed a straight line.

The rate of change seen in the table can also be seen on the graph.

- From the point for the pair of coordinates (3, 12), a move 1 unit to the right and 4 units up will result in the location of (4, 16).
- A subsequent move 1 unit to the right and 4 units up will result in the location of (5, 20).

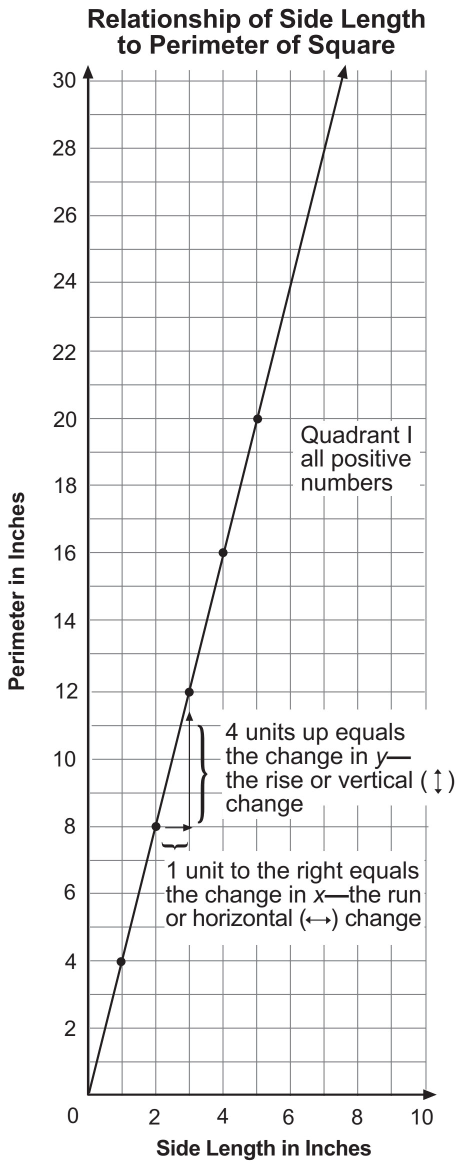
Slope—Vertical Change Compared to Horizontal Change

The vertical ($\updownarrow$) change compared to horizontal ($\longleftrightarrow$) change is called the **slope** of the line. The *slope* of a line is steepness or incline of a line. The slope of a line is usually seen as a **ratio** comparing the *vertical change to the horizontal change* or as a *ratio* comparing the *rise* (vertical change) to the *run* (horizontal change).

$$\text{slope} = \frac{\text{change in } y}{\text{change in } x} = \frac{\text{rise or vertical } (\updownarrow) \text{ change}}{\text{run or horizontal } (\longleftrightarrow) \text{ change}}$$

A *ratio* is the **quotient** of two numbers used to compare quantities.

In this instance the ratio is 4 to 1 or 4:1. See the graph again below.



The points are located in Quadrant I because the side lengths and perimeters are *positive numbers*.

To determine the perimeter of a square with a side length of 6 units, you could use the equation, the table, or the graph. Consider the advantages and disadvantages of each.

To determine the length of a side of a square with a perimeter of 28 units, you could use the equation, the table, or the graph. Consider the advantages and disadvantages of each.

Practice

Complete the following.

When running the 5000-meter race, a runner averaged 6 meters per second.

Equation: $d = 6s$

1. If d represents distance,

   s represents the number of ______________________ .

2. Complete the following table.

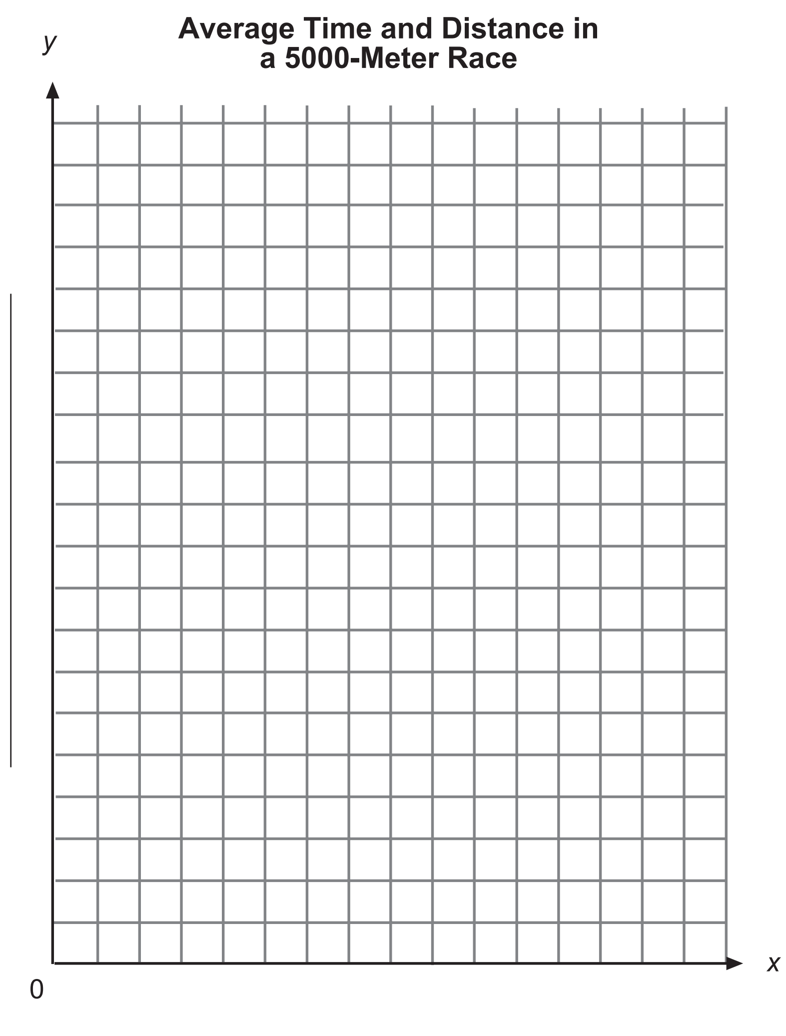
Average Time and Distance in a 5000-Meter Race

Time in Seconds s	Distance in Meters $d = 6s$
0	0
100	600
200	1200
300	1800
400	
500	
600	
700	
800	
s	$6s$

Note: Metric numbers with four digits are presented without a comma (e.g., 9960 meters).

3. On the grid provided, plot the coordinates from the previous table. The title has been provided.

- Choose an appropriate scale for each axis.
- Label each of your axes. (Since we are considering how *distance* traveled changes over *time*, time should be represented on the x-axis and distance on the y-axis.)



Note: Since the runner is averaging 6 meters per second you may choose to connect the points with a line. You should realize, however, that during a 100-second period in the actual race, the rate may have varied from 0 for a brief rest to more than 6 meters per second. The line is simply showing the overall picture of the race.

4. Summary of this problem is as follows:

   a. The equation, $d = 6s$, establishes a relationship between the

      ________________________ and the ________________________ .

   b. The table indicates that as the number of seconds increases by 100, the distance increases by ________________________ meters.

   c. Therefore, as the number of seconds increases by 1, the distance will increase by ________________________ meters.

   d. The connected points are linear, which means they form a straight line. The ratio describing the slope would be for each vertical change of ________________________ , there is a corresponding horizontal change of

      ________________________ .

5. a. What distance would this runner cover in 650 seconds?

      ____

   b. Did you use the equation, the table, or the graph to answer the question above? ____

   c. What was the basis for your choice? ________________________

      ____

      ____

6. a. How long would it take this runner to cover 2700 meters?

____

b. Did you use the equation, the table, or the graph to answer the question above? ____________________________________

c. What was the basis for your choice? ______________________

____

____

Practice

Complete the following.

A formula for changing temperature in degrees Celsius to temperature in degrees Fahrenheit is $F = 1.8C + 32$.

1. If F represents temperature in degrees Fahrenheit,

   C represents temperature in degrees ________________________ .

2. Complete the following table.

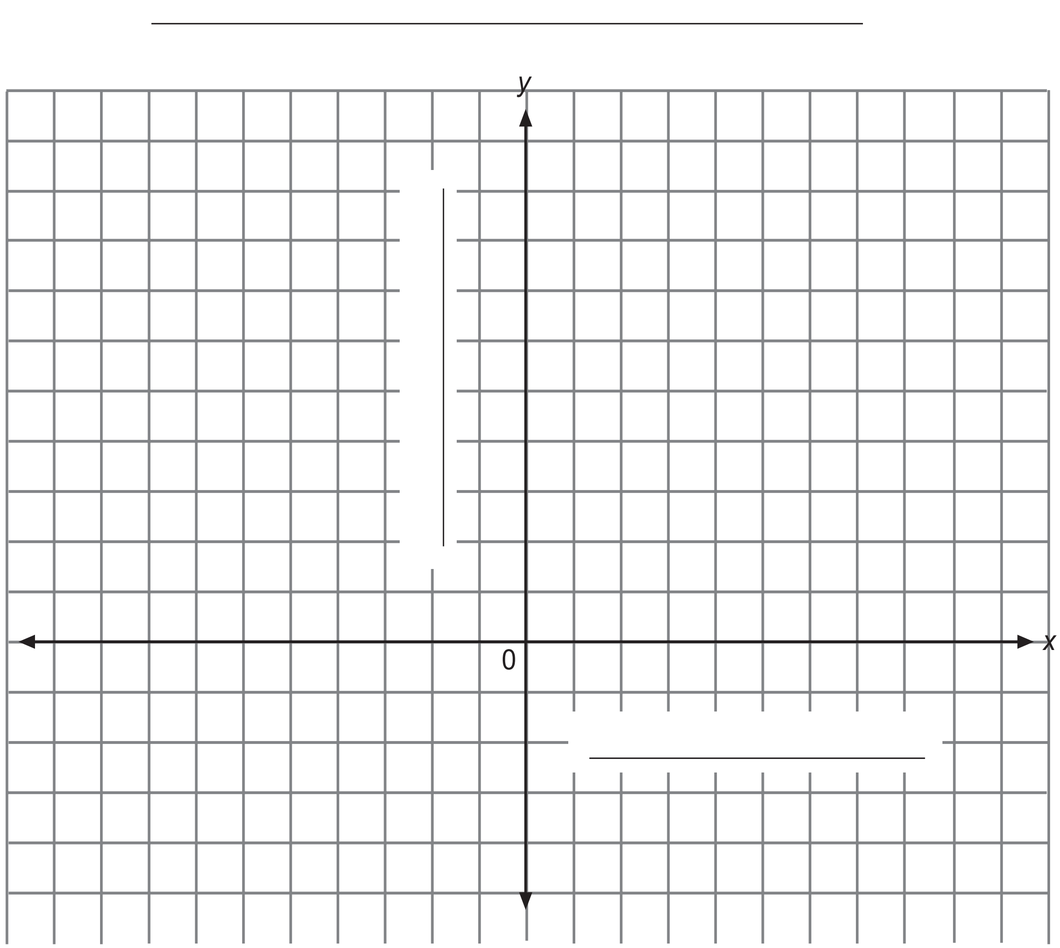
Relationship of Temperature in Degrees Celsius and Fahrenheit

Temperature in Degrees Celsius (C)	Temperature in Degrees Fahrenheit (F)
-20	-4
-10	
0	32
10	
20	
30	
C	$1.8C + 32$

3. On the grid provided, plot the coordinates from the previous table. Your graph should have the following:

- appropriate title
- labels for the axes
- appropriate scale.

Note: Since some temperatures are *negative*, you will need to show more than Quadrant I.



4. The summary for this problem is as follows:

   a. The equation, $F = 1.8C + 32$, establishes a relationship between the ______________ and ______________ . It tells us to find the **product** of 1.8 and C (the number of degrees Celsius) and then add 32.

   b. The table indicates that as the number of degrees Celsius increases by 10, the number of degrees Fahrenheit increases by ______________ . Therefore, as the number of degrees Celsius increases by 1, the number of degrees Fahrenheit increases by ______________ . The constant rate of change is ______________ .

   c. When plotted, the points are linear, which means they form a straight line. The ratio describing the slope would be for each vertical change of ______________ , there is a corresponding horizontal change of ______________ .

   d. The line ______________ (does, does not) pass through the origin).

   e. The line passes through all four quadrants *except* Quadrant ______________ .

   f. When the Celsius temperature is positive, the Fahrenheit temperature is ______________ (always, sometimes, never) positive.

5. a. What temperature in Fahrenheit would correspond to 15 degrees Celsius? ______________________

b. Did you use the equation, the table, or the graph to answer the question above? ______________________

c. What was the basis for your choice? ______________________

____

____

6. a. What temperature in Celsius would correspond to 77 degrees Fahrenheit? ______________________

b. Did you use the equation, the table, or the graph to answer the question above? ______________________

c. What was the basis for your choice? ______________________

____

____

Practice

Complete the following.

Forensic scientists use a formula to approximate the **height (*h*)** *of a female* if the *length of the humerus* (bone from shoulder to elbow) is known.

The formula is

$$h = 64.977 + 3.144H$$

where h represents the *height* in centimeters (cm) *of the female* and

H represents the length of the female's humerus in centimeters.

The formula used to approximate the *height of a male* if the *length of his humerus* is known is

$$h = 73.570 + 2.970H.$$

1. Complete the following table for females and males.

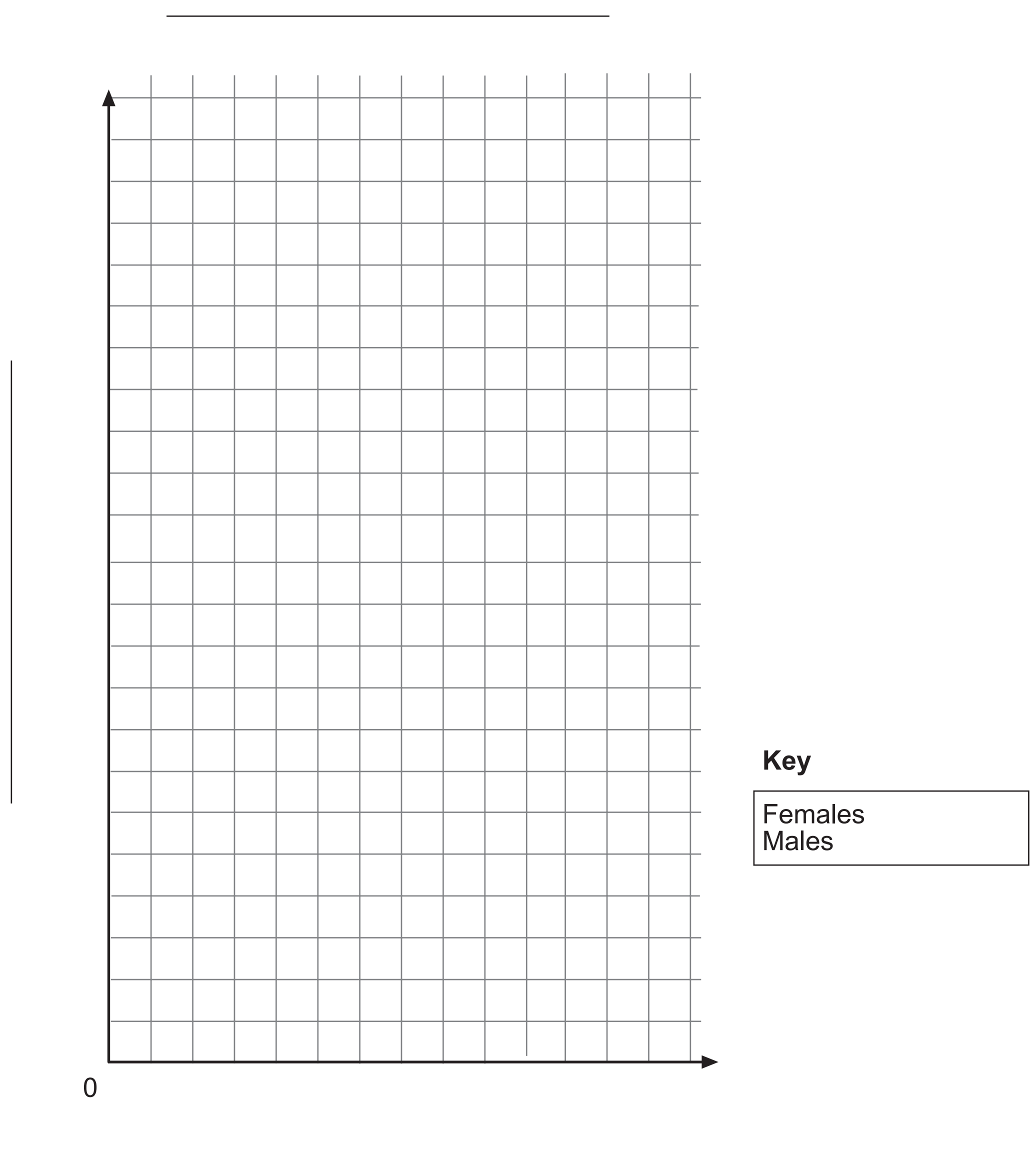
Relationship of Length of Humerus and Height

Length of Female's Humerus in Centimeters	Approximate Height of Female in Centimeters	Length of Male's Humerus in Centimeters	Approximate Height of Male in Centimeters
14	108.993	14	115.15
16	115.281	16	121.09
18	121.569	18	127.03
20		20	
22		22	
24		24	
H	$64.977 + 3.144H$	H	$73.570 + 2.970H$

2. On the grid provided, plot the coordinates from the previous table. Your graph should have the following:

- appropriate title
- labels for the axes
- appropriate scale.

Use different colors *or* symbols to plot the points for females and males.



3. Summary for this problem is as follows:

   a. The equations, $h = 64.977 + 3.144H$ and $h = 73.570 + 2.970H$, establish a relationship between the ________________________ and the ________________________ in females and males.

   b. The table indicates that as the length of the humerus in the female increases by 2 centimeters, height increases by ________________________ centimeters. The table also indicates that as the length of the humerus in the male increases by 2 centimeters, height increases by ________________________ centimeters.

   c. The points for females are linear, as are the points for males. The line for females is ________________________ (more, less) steep than the line for males. The constant rate of change for females is ________________________ (greater than, less than) the constant rate of change for males as reflected in the table and on the graphs.

4. a. What height for a female would correspond to a humerus of length 23 centimeters? ____________________

b. Did you use the equation, the table, or the graph to answer the question above? ____________________

c. What was the basis for your choice? ____________________

____

____

5. a. What length of humerus would correspond to a height of 189.4 centimeters for a male? ____________________

b. Did you use the equation, the table, or the graph to answer the question above? ____________________

c. What was the basis for your choice? ____________________

____

____

Characteristics of Some Linear Graphs

Each situation considered in this section had the following special characteristics.

- As the value for the first variable in the table increased at a constant rate, the values for the second variable in the table increased at a constant rate.
- In the equation, $y = mx$ or $y = mx + b$, this constant rate of change shows up as the **coefficient** of the variable x.

  Remember: The *coefficient* is the number part in front of the algebraic term signifying multiplication.

  Example:
  In $2x + 1$,
  2 is the coefficient.

- The points on the graph are linear.
- The constant rate of change shows up in the graph as the slope, a ratio of vertical change (change in y values) to horizontal change (change in x values). The greater the constant rate of change, the steeper the line.

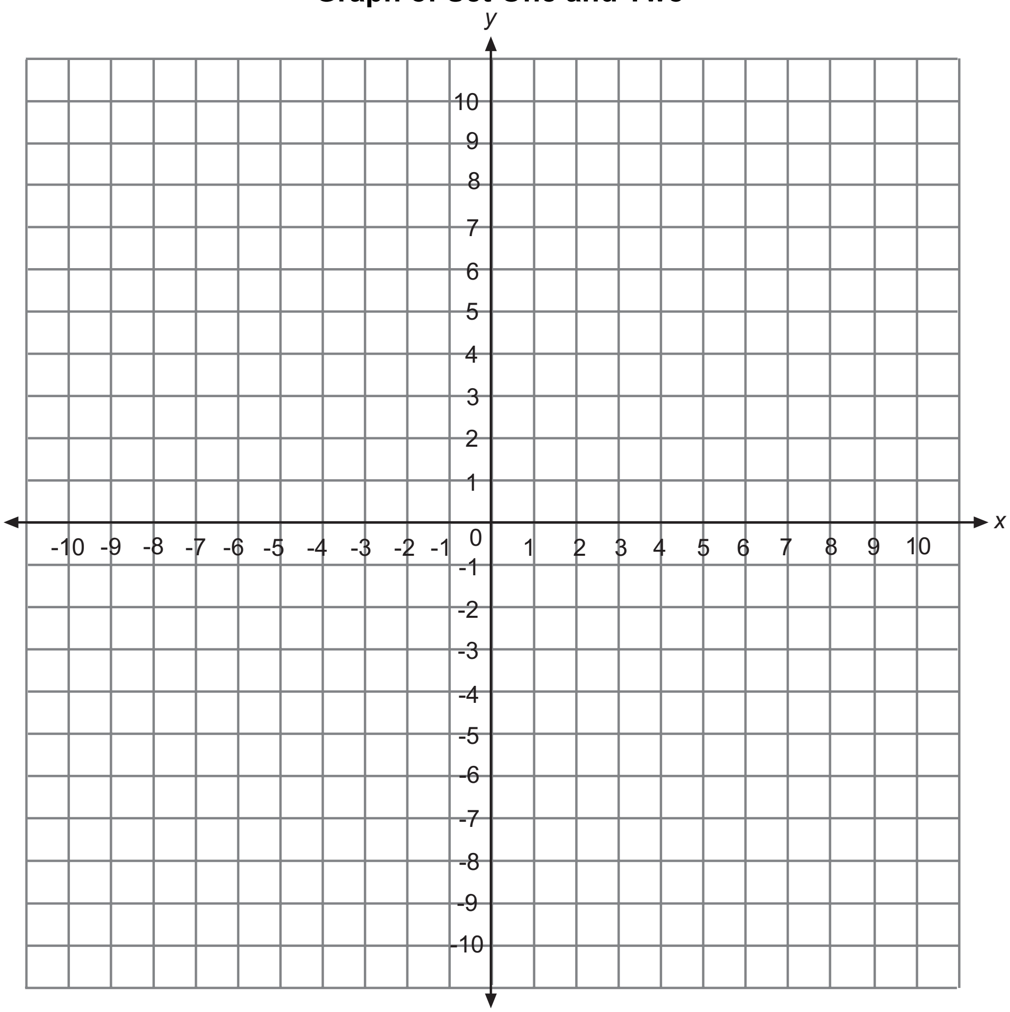
Practice

Complete the following.

1. a. On the grid provided, plot the following set of points. Use different colors *or* symbols to plot the points for *Set One* and *Set Two*. Connect the points in each set.

Set One: (4, 6), (5, 7), (6, 8)
Set Two: (2, 3), (4, 5), (6, 7)

Graph of Set One and Two



Key

Set One
Set Two

b. In *Set One*, as the *y*-coordinate increases 1 unit, the *x*-coordinate increases 1 unit. The slope would be represented by the ratio of 1:1. For each vertical increase of 1, there is a horizontal increase of 1.

In *Set Two*, as the *y*-coordinate increases 2 units, the

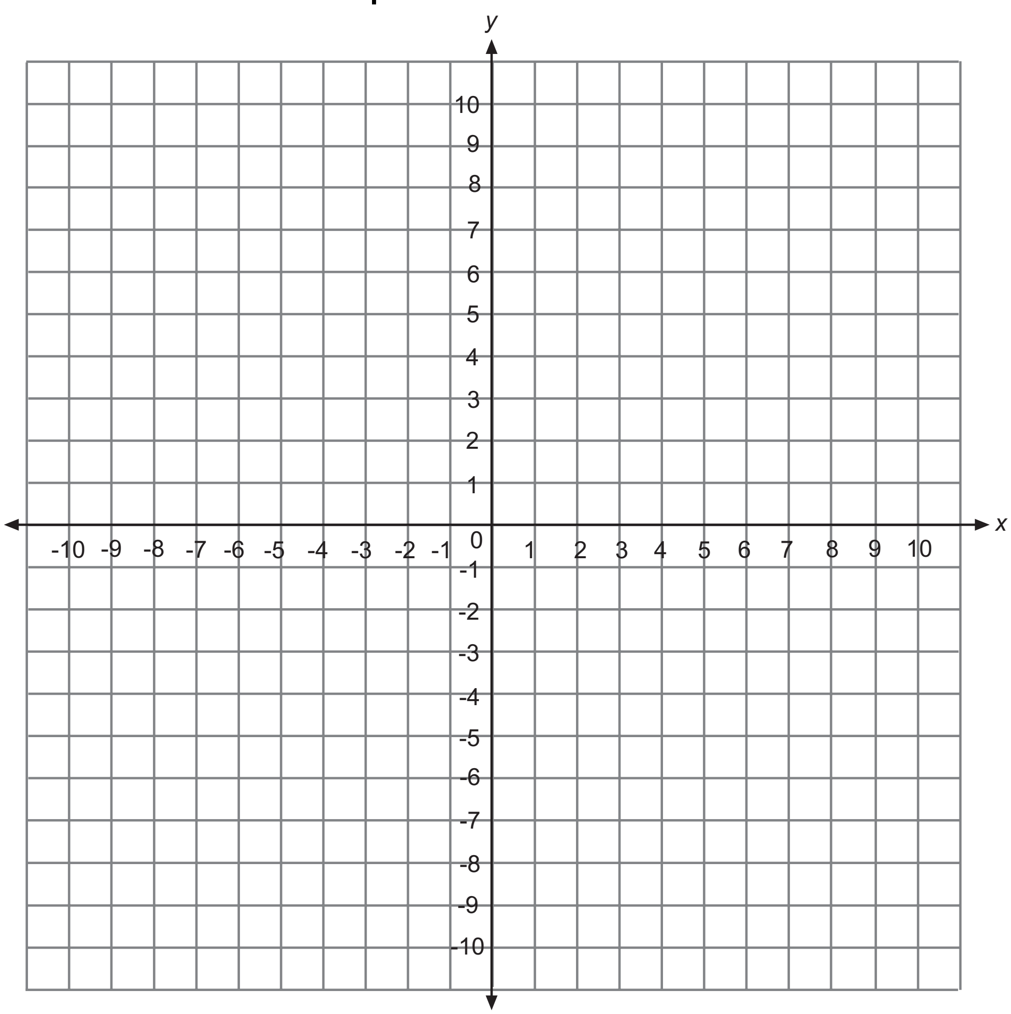
x-coordinate increases ____________ units. The slope would be represented by the ratio of 2:2. For each vertical increase of 2, there is a horizontal increase of 2.

Note: The ratios 1:1 and 2:2 are equivalent and the slopes are the same. The lines are therefore parallel.

2. a. On the grid provided, plot the following set of points. Use different colors *or* symbols to plot the points for *Set One* and *Set Two*. Connect the points in each set.

Set One: (-2, -5), (0, 0), (2, 5)
Set Two: (-1, -3), (3, 2), (7, 7)

Graph of Set One and Two



Key

Set One
Set Two

b. In *Set One*, as the y-values increase by 5, the x-values increase by

________________________.

c. The slope for this line can be represented as the ratio of

________________________ (vertical change) to

________________________ (horizontal change).

d. In *Set Two*, as the y-values increase by 5, the x-values increase by

________________________ .

e. The slope for this line can be represented as the ratio of

________________________ (vertical change) to

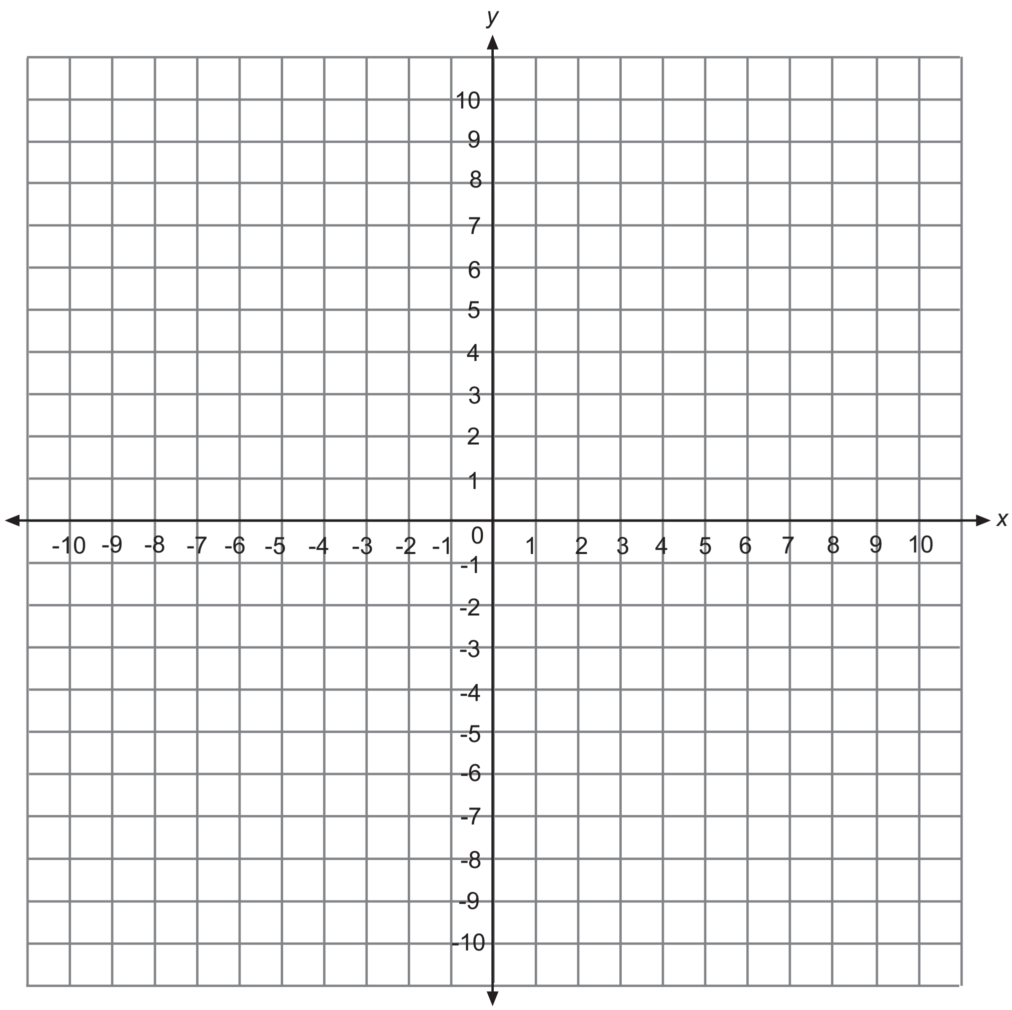
________________________ (horizontal change).

Note: The ratios 5:2 and 5:4 are not equivalent, so the slopes are not the same. The lines are therefore *not* parallel.

3. a. On the grid provided, plot the following set of points. Use different colors *or* symbols to plot the points for *Set One* and *Set Two*. Connect the points in each set.

Set One: (-2, 0), (0, 1), (2, 2)
Set Two: (-5, 1), (-1, 3), (3, 5)

Graph of Set One and Two



Key

Set One
Set Two

b. The slope for the line containing points from *Set One* can be represented by the ratio of __________ to __________ .

c. The slope for the line containing points from *Set Two* can be represented by the ratio of __________ to __________ .

d. These ratios ____________________ (are, are not) equivalent. The lines ____________________ (are, are not) parallel.

Practice

Use the list below to write the correct term for each definition on the line provided.

coefficient	**pattern or relationship**	**quotient**	**ratio**
height (h)	**perimeter (P)**	**rate of change**	**slope**
length (l)	**product**		

________________________ 1. the distance around a polygon

________________________ 2. the ratio of change in the vertical axis (y-axis) to each unit change in the horizontal axis (x-axis) in the form $\frac{\text{rise}}{\text{run}}$ or $\frac{\Delta y}{\Delta x}$

________________________ 3. how a quantity is changing over time

________________________ 4. a one-dimensional measure that is the measurable property of line segments

________________________ 5. the comparison of two quantities

________________________ 6. a line segment extending from the vertex or *apex* (highest point) of a figure to its base and forming a right angle with the base or plane that contains the base

________________________ 7. a predictable or prescribed sequence of numbers, objects, etc.

________________________ 8. the result of multiplying numbers together

________________________ 9. the result of dividing two numbers

________________________ 10. the number part in front of an algebraic term signifying multiplication

Practice

Match each definition with the correct term. Write the letter on the line provided.

______	1. any of four regions formed by the axes in a rectangular coordinate system	A. coordinate grid or plane
______	2. a transformation in which every point in a figure is moved in the same direction and by the same distance; also called a *slide*	B. graph
______	3. numbers greater than zero	C. negative numbers
______	4. a transformation that produces the mirror image of a geometric figure over a line of reflection; also called a *flip*	D. positive numbers
______	5. the vertical number line on a rectangular coordinate system	E. quadrant
______	6. numbers less than zero	F. reflection
______	7. the horizontal number line on a rectangular coordinate system	G. translation
______	8. the point common to the two rays that form an angle	H. value (of a variable)
______	9. a two-dimensional network of horizontal and vertical lines that are parallel and evenly-spaced	I. vertex
______	10. any of the numbers represented by the variable	J. x-axis
______	11. a drawing used to represent data	K. y-axis