Unit 5: Algebraic Thinking

This unit emphasizes strategies used to write algebraic expressions or equations and how to apply properties of algebra to solve equations.

Unit Focus

Number Sense, Concepts, and Operations

- Understand concrete and symbolic representations of real numbers in real-world situations. (MA.A.1.4.3)
- Understand and use the real number system. (MA.A.2.4.2)
- Select and justify alternative strategies, such as using properties of numbers, including inverse, identity, distributive, associative, and transitive, that allow operational shortcuts for computational procedures in real-world or mathematical problems. (MA.A.3.4.2)
- Add, subtract, multiply, and divide real numbers, including square roots and exponents, using appropriate methods of computing, such as mental mathematics, paper and pencil, and calculator. (MA.A.3.4.3)
- Use estimation strategies in complex situations to predict results and to check the reasonableness of results. (MA.A.4.4.1)

Measurement

- Use concrete and graphic models to derive formulas for finding rate, distance, time, and angle measurements. (MA.B.1.4.2)
- Solve real-world problems involving rated measures (miles per hour, feet per second). (MA.B.2.4.2)

- Solve real-world and mathematical problems involving estimates of measurements, including length, time, weight/mass, temperature, money, perimeter, area, and volume and estimate the effects of measurement errors on calculations. (MA.B.3.4.1)

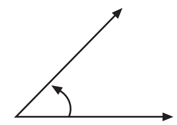
Algebraic Thinking

- Describe, analyze, and generalize relationships, patterns, and functions using words, symbols, variables, tables, and graphs. (MA.D.1.4.1)
- Represent real-world problem situations using finite graphs. (MA.D.2.4.1)
- Use equations and inequalities to solve real-world problems graphically and algebraically. (MA.D.2.4.2)

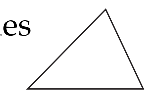
Vocabulary

Use the vocabulary words and definitions below as a reference for this unit.

acute angle .... an angle that measures less than 90° and greater than 0°



acute triangle .... a triangle with three acute angles

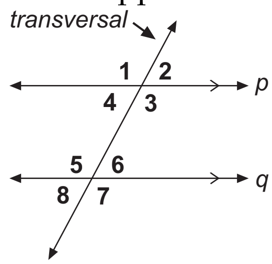


addend .... any number being added
Example: In 14 + 6 = 20,
14 and 6 are addends.

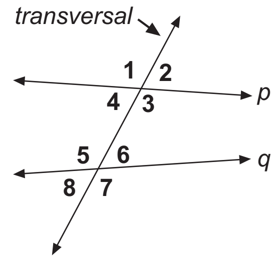
addition property of equality ..... adding the same number to each side of an equation results in an equivalent equation; for any real numbers a, b, and c, if $a = b$, then $a + c = b + c$

additive inverses .... a number and its opposite whose sum is zero (0); also called *opposites*
Example: In the equation 3 + -3 = 0,
3 and -3 are additive inverses, or *opposites*, of each other.

alternate angles .... a pair of angles that lie on opposite sides and at opposite ends of a transversal



Alternate angles are equal when the lines intersected by a transversal are parallel.



Even when lines cut by a transversal are *not* parallel, we still use the same vocabulary.

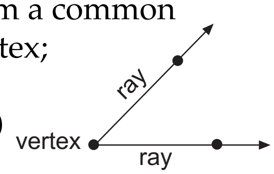
alternate exterior angles are angles whose points lie on the opposite sides of a transversal line and on the *outside* of the lines it intersects

∠1 and ∠7
∠2 and ∠8

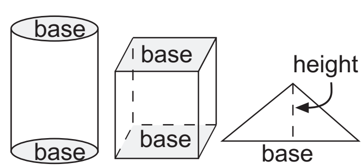
alternate interior angles are angles whose points lie on the opposite sides of a transversal line and on the *inside* of the lines it intersects

∠3 and ∠5
∠4 and ∠6

angle (∠) .... two rays extending from a common endpoint called the vertex; measures of angles are described in degrees (°)

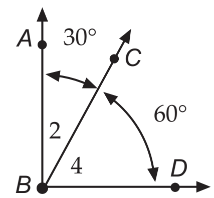


base (*b*) (geometric) .... the line or plane of a geometric figure, from which an altitude can be constructed, upon which a figure is thought to rest



commutative property................. the order in which two numbers are added or multiplied does *not* change their sum or product, respectively
Example: 2 + 3 = 3 + 2 *or* 4 x 7 = 7 x 4

complementary angles .... two angles, with measures the sum of which is exactly 90°

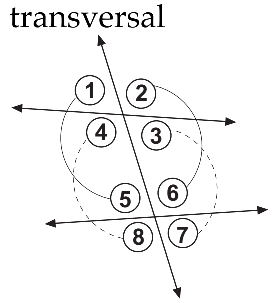


m∠*ABC* + m∠*CBD* = 90°
complementary angles

congruent (≅) .... figures or objects that are the same shape and size

consecutive .... in order
Example: 6, 7, 8 are consecutive whole numbers and 4, 6, 8 are consecutive even numbers.

corresponding angles .... a pair of angles that are in matching positions and lie on the same side of a transversal



Above are four pairs of corresponding angles; ∠1 and ∠5; ∠2 and ∠6; ∠3 and ∠7; and ∠4 and ∠8.

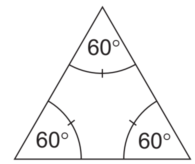
corresponding angles and sides .... the matching angles and sides in similar figures

degree (°) .... common unit used in measuring angles

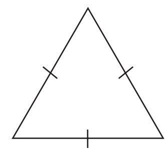
division property of equality ..... dividing the same number on each side of an equation results in an equivalent equation; for any real numbers *a*, *b*, and *c*, if $a = b$, and $c \neq 0$, then $\frac{a}{c} = \frac{b}{c}$

equation .... a mathematical sentence in which two expressions are connected by an equality symbol
Example: $2x = 10$

equiangular triangle .... a triangle with three equal angles



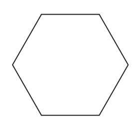
equilateral triangle .... a triangle with three congruent sides



even number .... any whole number divisible by 2
Example: 2, 4, 6, 8, 10, 12 …

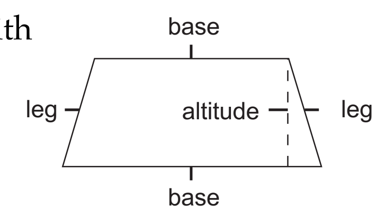
expression .... a collection of numbers, symbols, and/or operation signs that stands for a number
Example: $4r^2$; $3x + 2y$; $\sqrt{25}$
Expressions do *not* contain equality (=) or inequality (<, >, ≤, ≥, or ≠) symbols.

hexagon .... a polygon with six sides

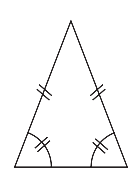


integers .... the numbers in the set
{… , -4, -3, -2, -1, 0, 1, 2, 3, 4, …}

isosceles trapezoid .... a trapezoid with congruent legs and two pairs of congruent base angles



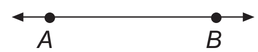
isosceles triangle .... a triangle with two congruent sides and two congruent angles



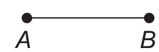
length (*l*) .... a one-dimensional measure that is the measurable property of line segments

like terms .... terms that have the same variables and the same corresponding exponents
Example: In $5x^2 + 3x^2 + 6$, $5x^2$ and $3x^2$ are like terms

line (↔) .... a collection of an infinite number of points in a straight pathway with unlimited length and having no width



line segment (—) .... a portion of a line that consists of two defined endpoints and all the points in between
Example: The line segment *AB* is between point *A* and point *B* and includes point *A* and point *B*.



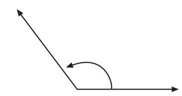
measure (m) of an angle (∠) .... the number of degrees (°) of an angle

multiples .... the numbers that result from multiplying a given whole number by the set of whole numbers
Examples: The multiples of 15 are 0, 15, 30, 45, 60, 75, etc.

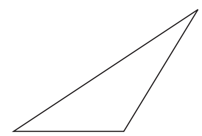
multiplication property of equality .... multiplying the same number on each side of an equation results in an equivalent equation; for any real numbers *a*, *b*, and *c*, if $a = b$, then $ac = bc$

multiplicative inverse (reciprocal) .... any two numbers with a product of 1
Example: 4 and $\frac{1}{4}$; zero (0) has no multiplicative inverse

obtuse angle .... an angle with a measure of more than 90° but less than 180°



obtuse triangle .... a triangle with one obtuse angle



odd number .... any whole number *not* divisible by 2
Example: 1, 3, 5, 7, 9, 11 …

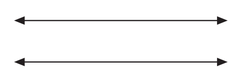
opposites .... two numbers whose sum is zero
Example:

$$-5 + 5 = 0 \quad \text{or} \quad \frac{2}{3} + \left(-\frac{2}{3}\right) = 0$$

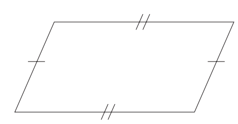
(-5 and 5 are opposites; $\frac{2}{3}$ and $-\frac{2}{3}$ are opposites)

parallel (||) .... being an equal distance at every point so as to never intersect

parallel lines .... two lines in the same plane that are a constant distance apart; lines with equal slopes



parallelogram .... a quadrilateral with two pairs of parallel sides

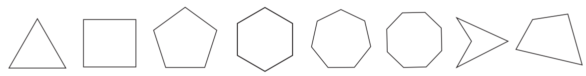


perimeter (*P*) .... the distance around a polygon

plane .... an infinite, two-dimensional geometric surface defined by three non-linear points or two distinct parallel or intersecting lines

point .... a specific location in space that has no discernable length or width

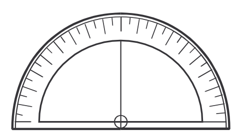
polygon .... a closed-plane figure, having at least three sides that are line segments and are connected at their endpoints
Example: triangle (3 sides), quadrilateral (4 sides), pentagon (5 sides), hexagon (6 sides), heptagon (7 sides), octagon (8 sides); concave, convex



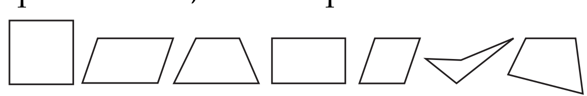
product .... the result of multiplying numbers together
Example: In 6 x 8 = 48, 48 is the product.

proportion .... a mathematical sentence stating that two ratios are equal
Example: The ratio of 1 to 4 equals 25 to 100, that is $\frac{1}{4} = \frac{25}{100}$

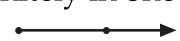
protractor .... an instrument used for measuring and drawing angles



quadrilateral .... polygon with four sides
Example: square, parallelogram, trapezoid, rectangle, rhombus, concave quadrilateral, convex quadrilateral



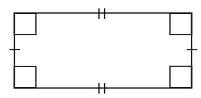
ray (**$\rightarrow$**) .... a portion of a line that begins at an endpoint and goes on indefinitely in one direction



real numbers .... the set of all rational and irrational numbers

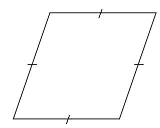
reciprocals .... two numbers whose product is 1
Example: Since $\frac{3}{4} \times \frac{4}{3} = 1$, the reciprocal of $\frac{3}{4}$ is $\frac{4}{3}$.

rectangle .... a parallelogram with four right angles

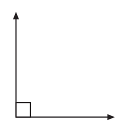


regular polygon .... a polygon that is both *equilateral* (all sides congruent) and *equiangular* (all angles congruent)

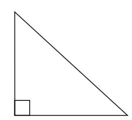
rhombus .... a parallelogram with four congruent sides



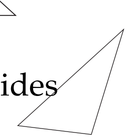
right angle .... an angle whose measure is exactly 90°



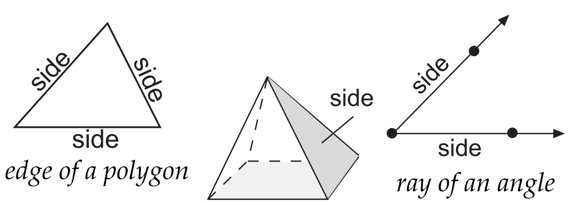
right triangle .... a triangle with one right angle



scalene triangle .... a triangle having no congruent sides



side .... the edge of a polygon, the face of a polyhedron, or one of the rays that make up an angle
Example: A triangle has three sides.



edge of a polygon
face of a polyhedron
ray of an angle

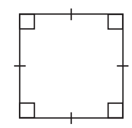
similar figures (~) .... figures that are the same shape, have corresponding, congruent angles, and have corresponding sides that are proportional in length

simplify an expression .... to perform as many of the indicated operations as possible

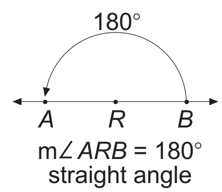
solution .... any value for a variable that makes an equation or inequality a true statement
Example: In $y = 8 + 9$
$y = 17$ 17 is the solution.

solve .... to find all numbers that make an equation or inequality true

square .... a rectangle with four sides the same length



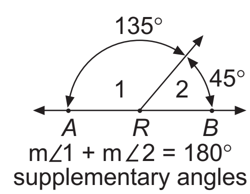
straight angle .... an angle that measures exactly 180°



subtraction property of equality .... subtracting the same number from each side of an equation results in an equivalent equation; for any real numbers a, b, and c, if $a = b$, then $a - c = b - c$

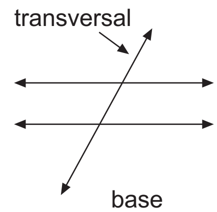
sum .... the result of adding numbers together
Example: In $6 + 8 = 14$,
14 is the sum.

supplementary angles .... two angles, with measures the sum of which is exactly 180°

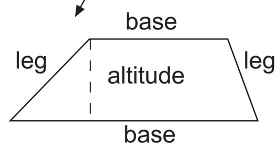


tessellation .... a covering of a plane with congruent copies of the same pattern with no holes and no overlaps
Example: floor tiles

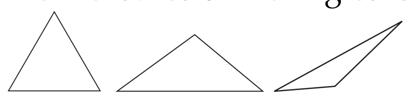
transversal .... a line that intersects two or more lines at different points



trapezoid .... a quadrilateral with just one pair of opposite sides parallel



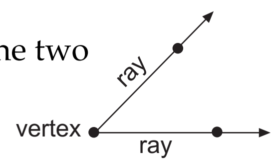
triangle .... a polygon with three sides; the sum of the measures of the angles is 180°



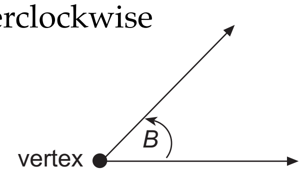
value (of a variable) .... any of the numbers represented by the variable

variable .... any symbol, usually a letter, which could represent a number

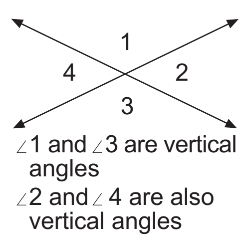
vertex .... the point common to the two rays that form an angle; the point common to any two sides of a polygon; the point common to three or more edges of a polyhedron; (plural: *vertices*); vertices are named clockwise or counterclockwise



vertex angle .... the point about which an angle is measured; the angle associated with a given vertex



vertical angles .... the opposite or non-adjacent angles formed when two lines intersect



Unit 5: Algebraic Thinking

Introduction

It is often helpful to be able to write algebraic expressions or equations from descriptions in words. It is also helpful to apply some properties of algebra as we solve equations. We will work in these areas in this chapter.

Lesson One Purpose

- Understand concrete and symbolic representations of real numbers in real-world situations. (MA.A.1.4.3)
- Understand and use the real number system. (MA.A.2.4.2)
- Select and justify alternative strategies, such as using properties of numbers, including inverse, identity, distributive, associative, and transitive, that allow operational shortcuts for computational procedures in real-world or mathematical problems. (MA.A.3.4.2)
- Add, subtract, multiply, and divide real numbers, including square roots and exponents, using appropriate methods of computing, such as mental mathematics, paper and pencil, and calculator. (MA.A.3.4.3)
- Use estimation strategies in complex situations to predict results and to check the reasonableness of results. (MA.A.4.4.1)
- Solve real-world and mathematical problems involving estimates of measurements, including length, time, weight/mass, temperature, money, perimeter, area, and volume and estimate the effects of measurement errors on calculations. (MA.B.3.4.1)
- Describe, analyze, and generalize relationships, patterns, and functions using words, symbols, variables, tables, and graphs. (MA.D.1.4.1)
- Use equations and inequalities to solve real-world problems graphically and algebraically. (MA.D.2.4.2)

Solving Algebraic Equations

Linda, the writer of this problem, is one of three children in her family. Her sister is 28 months older and her brother is 27 months younger. Let's consider ways to represent the ages algebraically.

First Way

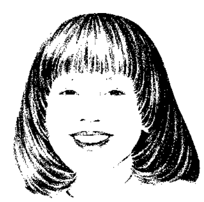
Sybil's age in months =	S
Linda's age in months =	$S - 28$
Jimmy's age in months =	$(S - 28) - 27$ *or* $S - 55$

Second Way

Jimmy's age in months =	J
Linda's age in months =	$J + 27$
Sybil's age in months =	$(J + 27) + 28$ *or* $J + 55$

Third Way

Linda's age in months =	L
Sybil's age in months =	$L + 28$
Jimmy's age in months =	$L - 27$


Sybil

Using the **equations** above, let's see how we can **solve** the following problem involving the ages of the children.

What were the ages of Linda and Jimmy when Sybil celebrated her 15th birthday?

Let's begin by using the *first way* described above because in this problem we know Sybil's age.

- Sybil's age in months = S, which for this problem is known to be 15 years x 12 (number of months in a year) = 180 months
- Linda's age in months = $S - 28 = 180 - 28 = 152 \div 12 = 12$ years 8 months

  Here's one way to figure it out.

  $$\begin{array}{r l} & 12 \text{ years} \\ 12\,\overline{)\,152} & \\ \underline{12} & \\ 32 & \\ \underline{24} & \\ 8 & \text{months} \end{array}$$

- Jimmy's age in months = $S - 55 = 180 - 55 = 125 \div 12 = 10$ years 5 months

Now let's consider the following *equation* representing the statement below and use the *third way* described on the previous page to solve a new problem.

> The **sum** of the ages of Sybil, Linda, and Jimmy is 577 months.
>
> $L + (L + 28) + (L - 27) = 577$
>
> where L represents Linda's age, $(L + 28)$ represents Sybil's age and $(L - 27)$ represents Jimmy's age.

What is Linda's age?

$L + (L + 28) + (L - 27) = 577$		
$L + L + 28 + L - 27 = 577$	←	commutative property
$L + L + L + 28 - 27 = 577$	←	combine like terms
$3L + 1 = 577$	←	
$3L + 1 \mathbf{+ (-1)} = 577 \mathbf{+ (-1)}$	←	addition property of equality; add -1 to each side
$3L = 576$	←	
$(3L)(\tfrac{1}{3}) = (576)(\tfrac{1}{3})$	←	multiplication property of equality; multiply each side by $\frac{1}{3}$
$L = 192$	←	


Linda

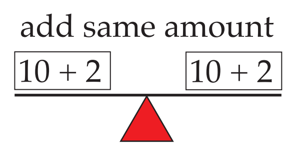
Therefore, Linda's age is 192 months. When divided by 12, which equals one year, Linda's age equals exactly 16 years.

Think about This!

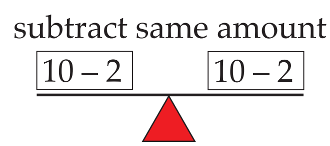
Some important properties were used in solving the equation on the previous page.

- The **commutative property** of addition allows the order of two **addends** to be changed. Note the 28 + *L* on the *second line* and the *L* + 28 on the *third line*.
- To **simplify an expression** you perform as many of the indicated operations as possible. One way to *simplify an expression* is to combine **like terms**. *Like terms* will have the same **variables**. Note how the *L* + *L* + *L* on the third line combine to become 3*L* on the fourth line.
- An equation is like a teeter-totter.

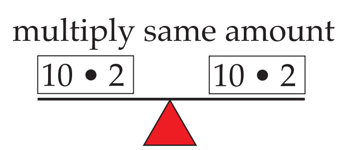
We can *add* the *same amount* to each side and it remains balanced.



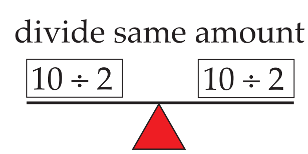
We can *take* the *same amount* from each side and it remains balanced.



We can *multiply* each side by the *same number* or



divide each side by the *same number* and it remains balanced.



Note how -1 is being added to each side on *line five*. Notice that -1 is the **additive inverse**, or the **opposite**, of 1. The **addition property of equality** allows this. Notice, also, how each side is being multiplied by $\frac{1}{3}$, the **multiplicative inverse**, or **reciprocal**, of 3 on *line seven*. The **multiplication property of equality** allows us to do this.

See the properties below.

Properties

Order (Commutative Property) The order in which any two numbers are added or multiplied does not change the **sum** or **product**.	
Commutative Property of Addition	**Commutative Property of Multiplication**
Numbers can be added in any order without changing the *sum*. For any **real numbers** a and b, $a + b = b + a$. $10 + 2 = 2 + 10$ $x + 2 = 2 + x$	Numbers can be multiplied in any order without changing the *product*. For any *real numbers* a and b, $a \bullet b = b \bullet a$. $10 \bullet 2 = 2 \bullet 10$ $x \bullet 2 = 2 \bullet x$
Equality Properties Equality properties keep an equation in balance.	
Addition Property of Equality	**Subtraction Property of Equality**
The same number can be added to each side of an equation and the equation remains balanced. For any real numbers a, b, and c, if $a = b$, then $a + c = b + c$.	The same number can be subtracted from each side of an equation and the equation remains balanced. For any real numbers a, b, and c, if $a = b$, then $a - c = b - c$.
Multiplication Property of Equality	**Division Property of Equality**
The same number can be multiplied on each side of an equation and the equation remains balanced. For any real numbers a, b, and c, if $a = b$, then $ac = bc$.	The same number can be divided on each side of an equation and the equation remains balanced. For any real numbers a, b, and c, if $a = b$, and $c \neq 0$, then $\frac{a}{c} = \frac{b}{c}$.

Practice

Solve the following.

1. The sum of Linda's age and Sybil's age, in months, when Linda began taking Algebra I was 374. What was Linda's age in months? What was Linda's age in years and months?

   Linda's age = L
   Sybil's age = $L + 28$

   Words to symbols:

   The sum of Linda's age and Sybil's age was 374.

   $$\underbrace{}_{L} + \underbrace{}_{L + 28} = 374$$

   Solving the equation:

   $L + L + 28 = 374$ ← combine like terms
   ________ $+ 28 = 374$
   ________ = ________ ← add -28 to each side
   ________ = ________ ← complete arithmetic on each side
   ________ = ________ ← multiply each side by _____
   ________ = ________

   Therefore, Linda's age in months when she began taking Algebra I

   was ________ months *or*

   ________ years, ________ months. (Refer to pages 272 and 273 as needed.)

   Remember: 12 months = 1 year

2. At the time of Linda's wedding, the sum of Linda's age and Jimmy's age, in months, was 471. How old was Jimmy in months? How old was Jimmy in years and months?

Jimmy

Jimmy's age = J
Linda's age = $J + 27$

The sum of Linda's age and Jimmy's age was 471.

$$J + 27 + J = 471$$

Solving the equation:

$J + 27 + J$	=	471	commutative property
______________	=	471	combine like terms
______________	=	471	
______________	=	471 + (-27)	add -27 to each side
______________	=	____________	complete arithmetic on each side
______________	=	____________	multiply each side by ____
______________	=	____________	

Therefore, Jimmy's age was ____________ months *or*

____________ years, ____________ months at the time of Linda's

wedding.

3. At the time Elvis Presley appeared on the stage in their hometown, the sum of the ages of Sybil and Linda was 370, in months. What was Sybil's age in months? What was Sybil's age in years and months?

Sybil's age =	S
Linda's age =	$S - 28$

Write an equation and solve the problem. Show all your work.

Sybil's age was ____________ months or ____________ years,

____________ months at the time Elvis Presley appeared on stage in her hometown.

4. You will notice that when Linda's age was being sought in problem 1, we used L for her age and stated Sybil's age in terms of Linda's, $L + 28$. ($L + L + 28 = 374$)

We could have done the problem in the following way:

Sybil's age = S
Linda's age = $S - 28$

$S + S - 28$	$=$	374	combine like terms
$2S - 28$	$=$	374	
$2S - 28 + 28$	$=$	$374 + 28$	add + 28 to each side
$2S$	$=$	402	complete arithmetic on each side
$(2S)(\frac{1}{2})$	$=$	$(402)(\frac{1}{2})$	multiply each side by $\frac{1}{2}$
S	$=$	201	

Sybil's age was 201 months.

Linda' age was $S - 28$, so Linda's age was $201 - 28$ or 173 months.

Which way do you prefer? Why?

____

____

____

____

5. When Jimmy's age was being sought in problem 2, we used J for his age and stated Linda's age in terms of Jimmy's, $J + 27$.

We could have solved it in the following way:

$L + L - 27 = 471$	combine like terms
$2L - 27 = 471$	
$2L - 27 + 27 = 471 + 27$	add + 27 to each side
$2L = 498$	complete arithmetic on each side
$(2L)(\frac{1}{2}) = (498)(\frac{1}{2})$	multiply each side by $\frac{1}{2}$
$L = 249$	

Linda's age was 249 months.

Jimmy's age was $L - 27$ or $249 - 27$ or 222 months.

Which way do you prefer? Why?

6. When Sybil's age was being sought in problem 3, we used S for her age and stated Linda's age in terms of Sybil's, $S - 28$.

We could have solved it in the following way:

$L + L + 28$	$=$	370	combine like terms
$2L + 28$	$=$	370	
$2L + 28 + (-28)$	$=$	$370 + (-28)$	add -28 to each side
$2L$	$=$	342	complete arithmetic on each side
$(2L)(\frac{1}{2})$	$=$	$(342)(\frac{1}{2})$	multiply each side by $\frac{1}{2}$
L	$=$	171	

Linda's age was 171 months.

Sybil's age was $L + 28$ or $171 + 28$ or 199 months.

Which do you prefer? Why?

____

____

____

____

7. At the time of Sybil's high school graduation, the sum of her age and Jimmy's age was 387 months. What was Sybil's age in months? What was Sybil's age in years and months?

Sybil's age = S
Jimmy's age = $(S - 28) - 27$ *or* $S - 55$

Write an equation and solve the problem. Show all your work.

8. At the time Linda's first grandchild was born, the sum of the ages of Sybil, Linda, and Jimmy was 1,873 months. Each of the following equations in 8a, 8b, and 8c would allow us to determine Linda's age at this time. Solve each of the equations to determine Linda's age. Show all your work.

Remember: For problems 8a and 8c, after solving the equation, use that **solution** to then find Linda's age.

a. Sybil's age in months $= S$
Linda's age in months $= S - 28$
Jimmy's age in months $= S - 55$

$S + (S - 28) + (S - 55) = 1{,}873$

b. Linda's age in months $= L$
Sybil's age in months $= L + 28$
Jimmy's age in months $= L - 27$

$(L + 28) + L + (L - 27) = 1{,}873$

c. Jimmy's age in months $= J$
Linda's age in months $= J + 27$
Sybil's age in months $= J + 55$

$J + (J + 27) + (J + 55) = 1{,}873$

d. State which type of equation you preferred to determine Linda's age and why.

____

____

____

9. George Herbert Walker Bush was 22 years and 1 month old at the time his son, George Walker Bush, was born. Each man has served as President of the United States. At the time George Herbert Walker Bush was inaugurated as president, the sum of their ages was 1,285 months. What was the age of the son at the time of his father's inauguration?

Write an equation and solve the problem. Show all your work.

Age in months of George Herbert Walker Bush = ________________

Age in months of George Walker Bush = ________________________

The sum of their ages at the time of George Herbert Walker Bush's

inauguration = ____________ + ____________ = 1,285 months

Now solve the equation.

George Walker Bush was ____________ months old, or

____________ years and ____________ months old at the time of his

father's inauguration as president.

10. George Walker Bush was 6 years 7 months old when his younger brother, Jeb, was born. At the time Jeb Bush was first inaugurated as the Governor of Florida, the sum of their ages was 1,205 months. What was the age of Jeb Bush at the time of his first inauguration as Governor of Florida?

Write an equation and solve the problem. Show all your work.

Age of Jeb Bush in months = ______________________________

Age of George Walker Bush in months = ____________________

The sum of their ages at the time of Jeb Bush's first inauguration as

Governor of Florida = __________ + __________ = 1,205 months

Now solve the equation.

The age of Jeb Bush at the time of his first inauguration as Governor

of Florida was __________ months or __________ years

__________ months.

11. Consider your age and that of other family members or friends. Write an interesting problem that would allow someone to use algebra to solve.

Practice

Match each definition with the correct term. Write the letter on the line provided.

______ 1. terms that have the same variables and the same corresponding exponents

______ 2. to find all numbers that make an equation or inequality true

______ 3. a number and its opposite whose sum is zero (0)

______ 4. the order in which two numbers are added or multiplied does *not* change their sum or product

______ 5. adding the same number to each side of an equation results in an equivalent equation

______ 6. to perform as many of the indicated operations as possible

______ 7. a mathematical sentence in which two expressions are connected by an equality symbol

______ 8. any number being added

______ 9. the result of adding numbers together

______ 10. any symbol, usually a letter, which could represent a number

______ 11. any two numbers with a product of 1

______ 12. multiplying the same number on each side of an equation results in an equivalent equation

A. addend

B. addition property of equality

C. additive inverses

D. commutative property

E. equation

F. like terms

G. multiplication property of equality

H. multiplicative inverse (reciprocal)

I. simplify an expression

J. solve

K. sum

L. variable