Unit 8: Working with Ratios and Proportions

This unit emphasizes the use of ratio and proportion in a variety of ways and provides a review of scientific notation.

Unit Focus

Number Sense, Concepts, and Operations

- Associate verbal names, written word names, and standard numerals with integers, rational numbers, irrational numbers, and real numbers. (MA.A.1.4.1)
- Understand concrete and symbolic representations of real numbers in real-world situations. (MA.A.1.4.3)
- Understand that numbers can be represented in a variety of equivalent forms, including integers, fractions, decimals, percents, scientific notation, exponents, radicals, and absolute value. (MA.A.1.4.4)
- Understand and use the real number system. (MA.A.2.4.2)
- Add, subtract, multiply, and divide real numbers, including square roots and exponents, using appropriate methods of computing, such as mental mathematics, paper and pencil, and calculator. (MA.A.3.4.3)
- Use estimation strategies in complex situations to predict results and to check the reasonableness of results. (MA.A.4.4.1)

Measurement

- Relate the concepts of measurement to similarity and proportionality in real-world situations. (MA.B.1.4.3)
- Select and use direct (measured) and indirect (not measured) methods of measurement as appropriate. (MA.B.2.4.1)

- Solve real-world problems involving rated measures (miles per hour, feet per second). (MA.B.2.4.2)
- Solve real-world and mathematical problems involving estimates of measurements, including length, time, weight/mass, temperature, money, perimeter, area, and volume and estimate the effects of measurement errors on calculations. (MA.B.3.4.1)

Geometry and Spatial Sense

- Understand geometric concepts such as perpendicularity, parallelism, congruency, similarity, and symmetry. (MA.C.2.4.1)

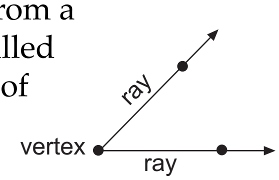
Algebraic Thinking

- Represent real-world problem situations using finite graphs. (MA.D.2.4.1)
- Use equations and inequalities to solve real-world problems graphically and algebraically. (MA.D.2.4.2)

Vocabulary

Use the vocabulary words and definitions below as a reference for this unit.

angle (∠) .... two rays extending from a common endpoint called the vertex; measures of angles are described in degrees (°)



area (*A*) .... the measure, in square units, of the inside region of a two-dimensional figure
Example: A rectangle with sides of 4 units by 6 units contains 24 square units or has an area of 24 square units.

congruent (≅) .... figures or objects that are the same shape and size

corresponding angles and sides .... the matching angles and sides in similar figures

cross product .... the product of one numerator and the opposite denominator in a pair of fractions
Example:

Is $\frac{2}{5}$ equal to $\frac{6}{15}$?

$\frac{2}{5} \stackrel{?}{=} \frac{6}{15}$

$2 \times 15 \stackrel{?}{=} 5 \times 6$ — The cross products are 2 x 15 and 5 x 6.

$30 = 30$ — Both cross products equal 30. The cross products of equivalent fractions are equal.

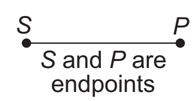
Yes, $\frac{2}{5} = \frac{6}{15}$.

decimal number .... any number written with a decimal point in the number
Example: A decimal number falls between two whole numbers, such as 1.5 falls between 1 and 2. Decimal numbers smaller than 1 are sometimes called decimal fractions, such as five-tenths is written 0.5.

degree (°) .... common unit used in measuring angles

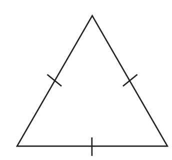
denominator .... the bottom number of a fraction, indicating the number of equal parts a whole was divided into
Example: In the fraction $\frac{2}{3}$ the denominator is 3, meaning the whole was divided into 3 equal parts.

endpoint .... either of two points marking the end of a line segment



equation .... a mathematical sentence in which two expressions are connected by an equality symbol
Example: $2x = 10$

equilateral triangle .... a triangle with three congruent sides



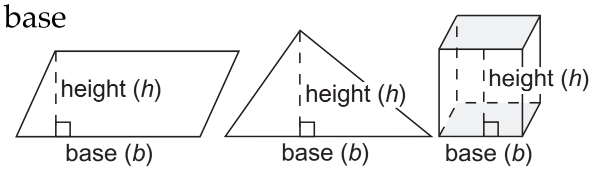
equivalent
(forms of a number)...................... the same number expressed in different forms
Example: $\frac{3}{4}$, 0.75, and 75%

exponent (exponential form) ...... the number of times the base occurs as a factor
Example: 2^3 is the exponential form of $2 \times 2 \times 2$. The numeral two (2) is called the *base*, and the numeral three (3) is called the *exponent*.

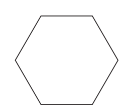
factor .... a number or expression that divides evenly into another number
Example: 1, 2, 4, 5, 10, and 20 are factors of 20 and $(x + 1)$ is one of the factors of $(x^2 - 1)$.

fraction .... any part of a whole
Example: One-half written in fractional form is $\frac{1}{2}$.

height (*h*) .... a line segment extending from the vertex or *apex* (highest point) of a figure to its base and forming a right angle with the base or plane that contains the base



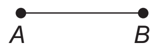
hexagon .... a polygon with six sides



least common multiple (LCM) .... the smallest of the common multiples of two or more numbers
Example: For 4 and 6, 12 is the least common multiple.

length (*l*) .... a one-dimensional measure that is the measurable property of line segments

line segment (—) .... a portion of a line that consists of two defined endpoints and all the points in between



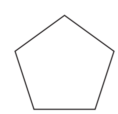
Example: The line segment *AB* is between point *A* and point *B* and includes point *A* and point *B*.

multiplicative identity .... the number one (1); the product of a number and the multiplicative identity is the number itself
Example: 5 x 1 = 5

multiplicative property of equality .... if $a = b$, then $ac = bc$; if you multiply (or divide) by the same number on both sides of an equation, the equation continues to be true

numerator .... the top number of a fraction, indicating the number of equal parts being considered
Example: In the fraction $\frac{2}{3}$, the numerator is 2.

pentagon .... a polygon with five sides

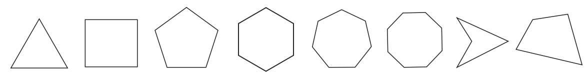


percent (%) .... a special-case ratio which compares numbers to 100 (the second term)
Example: 25% means the ratio of 25 to 100.

perimeter (*P*) .... the distance around a polygon

perpendicular (⊥) .... two lines, two line segments, or two planes that intersect to form a right angle

polygon .... a closed-plane figure, having at least three sides that are line segments and are connected at their endpoints
Example: triangle (3 sides), quadrilateral (4 sides), pentagon (5 sides), hexagon (6 sides), heptagon (7 sides), octagon (8 sides); concave, convex



power (of a number) .... an exponent; the number that tells how many times a number is used as a factor
Example: In 2^3, 3 is the power.

prime factorization .... writing a number as the product of prime numbers
Example: $24 = 2 \times 2 \times 2 \times 3 = 2^3 \times 3$

prime number .... any whole number with only two whole number factors, 1 and itself
Example: 2, 3, 5, 7, 11, etc.

product .... the result of multiplying numbers together
Example: In $6 \times 8 = 48$, 48 is the product.

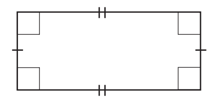
proportion .... a mathematical sentence stating that two ratios are equal
Example: The ratio of 1 to 4 equals 25 to 100, that is $\frac{1}{4} = \frac{25}{100}$.

proportional .... having the same or constant ratio
Example: Two quantities that have the same ratio are considered directly proportional.
If $y = kx$, then y is said to be directly proportional to x and the constant of proportionality is k.

quotient .... the result of dividing two numbers
Example: In $42 \div 7 = 6$,
6 is the quotient.

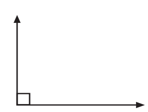
ratio .... the comparison of two quantities
Example: The ratio of a and b is a:b or $\frac{a}{b}$,
where $b \neq 0$.

rectangle .... a parallelogram with four right angles



regular polygon .... a polygon that is both *equilateral* (all sides congruent) and *equiangular* (all angles congruent)

right angle .... an angle whose measure is exactly 90°



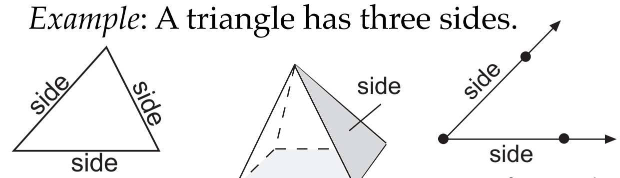
rounded number .... a number approximated to a specified place
Example: A commonly used rule to round a number is as follows.
- If the digit in the first place after the specified place is 5 or more, *round up* by adding 1 to the digit in the specified place (461 rounded to the nearest hundred is 500).
- If the digit in the first place after the specified place is less than 5, *round down* by *not* changing the digit in the specified place (441 rounded to the nearest hundred is 400).

scale .... the relationship between the measures on a drawing or model and the real object

scale factor .... the constant that is multiplied by the lengths of each side of a figure that produces an image that is the same shape as the original figure

scientific notation .... a shorthand method of writing very large or very small numbers using exponents in which a number is expressed as the product of a power of 10 and a number that is greater than or equal to one (1) and less than 10
Example: $7.59 \times 10^5 = 759{,}000$

side .... the edge of a polygon, the face of a polyhedron, or one of the rays that make up an angle
Example: A triangle has three sides.



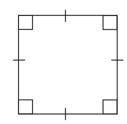
edge of a polygon *face of a polyhedron* *ray of an angle*

similar figures (~) .... figures that are the same shape, have corresponding, congruent angles, and have corresponding sides that are proportional in length

solution .... any value for a variable that makes an equation or inequality a true statement
Example: In $y = 8 + 9$
$y = 17$ 17 is the solution.

solve .... to find all numbers that make an equation or inequality true

square .... a rectangle with four sides the same length

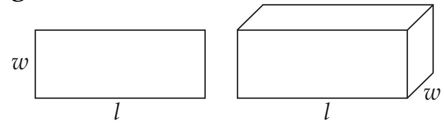


standard form .... a method of writing the common symbol for a numeral
Example: The standard numeral for five is 5.

sum .... the result of adding numbers together
Example: In $6 + 8 = 14$, 14 is the sum.

value (of a variable) .... any of the numbers represented by the variable

width (*w*) .... a one-dimensional measure of something side to side



Unit 8: Ratios and Proportions

Introduction

Ratios are used to help keep things in proportion. What might you need to keep in proportion?

- You might need to double your Aunt Selma's cookie recipe for tonight's dinner. By using twice as much of each ingredient, you would be making twice as many cookies, while keeping everything in proportion.
- You might need to figure out who was the best hitter in the last baseball season. Since some batters may get more chances to hit than others, using ratios and percents will help you make fair comparisons.

You can do more than change recipe amounts and make fair comparisons. By using ratios and percents in proportions, you can also make models and drawings.

Lesson One Purpose

- Associate verbal names, written word names, and standard numerals with integers, rational numbers, irrational numbers, and real numbers. (MA.A.1.4.1)
- Understand concrete and symbolic representations of real numbers in real-world situations. (MA.A.1.4.3)
- Understand that numbers can be represented in a variety of equivalent forms, including integers, fractions, decimals, percents, scientific notation, exponents, radicals, and absolute value. (MA.A.1.4.4)
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- Add, subtract, multiply, and divide real numbers, including square roots and exponents, using appropriate methods of computing, such as mental mathematics, paper and pencil, and calculator. (MA.A.3.4.3)

- Use estimation strategies in complex situations to predict results and to check the reasonableness of results. (MA.A.4.4.1)
- Relate the concepts of measurement to similarity and proportionality in real-world situations. (MA.B.1.4.3)
- Select and use direct (measured) and indirect (not measured) methods of measurement as appropriate. (MA.B.2.4.1)
- Solve real-world problems involving rated measures (miles per hour, feet per second). (MA.B.2.4.2)
- Solve real-world and mathematical problems involving estimates of measurements, including length, time, weight/mass, temperature, money, perimeter, area, and volume and estimate the effects of measurement errors on calculations. (MA.B.3.4.1)
- Understand geometric concepts such as perpendicularity, parallelism, congruency, similarity, and symmetry. (MA.C.2.4.1)
- Use equations and inequalities to solve real-world problems graphically and algebraically. (MA.D.2.4.2)

Ratios

A **ratio** is a comparison of two quantities that shows the **scale**, or relationship, between the measures. *Ratios* may be expressed as **quotients**, **fractions**, **decimals**, or **percents**, or in the form *a*:*b*.

Think about This!

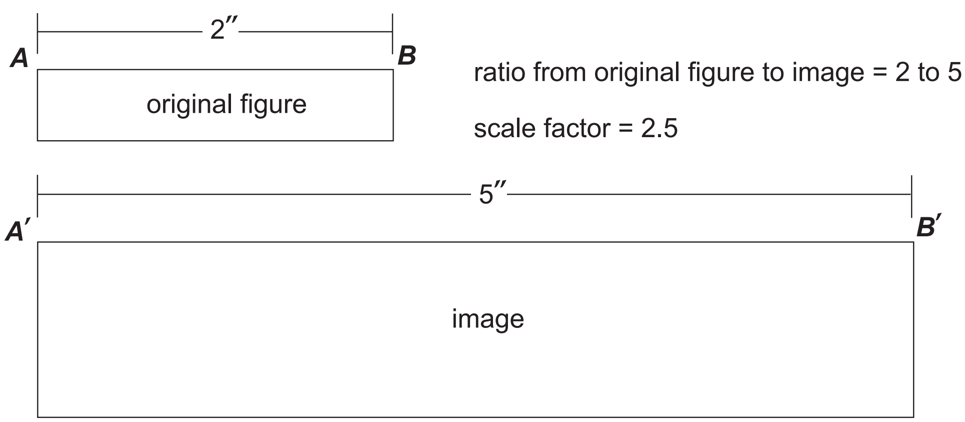
Sales tax is an example of ratio. In Tallahassee, Florida, residents paid 7.5 cents sales tax for 100 cents, or dollar, spent on goods that were subject to sales tax at the time this book was written. The ratio of sales tax to price of item is 7.5 to 100 or 0.075 to 1.00.

7.5 to 100 *or* 0.075 to 1.00	7.5:100 *or* 0.075:1.00	$\frac{7.5}{100}$ *or* $\frac{0.075}{1.00}$

ratio of sales tax to price expressed three different ways

Scale Factor

A **scale factor** between an original figure and its image is another example of ratio. For every 2 inches in the **length (*l*)** of **side** *AB* in the original figure, there will be 5 inches in the *length* of *side A'B'* in the image. The *scale factor* is said to be 2.5, while the ratio from original to image is 2 to 5.



When you write an **equation** stating that two ratios are equal, you are writing a **proportion**. You will explore several ways to **solve** such *equations*.

Practice

To complete the following, we will examine **three different methods** *to* **solve ratios** *and* **proportions***.*

1. A wallet costs $12. Sales tax of 7.5% is to be charged. To determine the amount of sales tax, a proportion can be written as follows:

$$\frac{0.075}{1.00} = \frac{x}{12.00}$$

Method One

a. Think about these as two **equivalent** fractions. What must the **denominator** (bottom number of the fraction) of 1.00 be multiplied by to get 12.00?

____

b. If the **numerator** (top number of the fraction) 0.075 is

multiplied by that same amount, the **value** of x is ____________ .

This method is based on the *identity property for multiplication*.

One (1) is the **multiplicative identity**.

We multiply the left side of the equation by a fraction *equivalent* to 1 that results in a **solution** for x.

$$\frac{0.075}{1.00} \bullet \frac{12}{12} = \frac{x}{12.00}$$

$$\frac{0.90}{12} = \frac{x}{12}$$

$$x = 0.90$$

Solve the following using **Method One**.

2. A figure is to be enlarged. For each 2 inches in the side length of the original, there will be 5 inches in the **corresponding side** (or matching sides in a similar figure) of the image. If one of the side lengths in the image is 13 inches, what was the length of the *corresponding side* in the original?

$$\frac{2}{5} = \frac{x}{13}$$

   a. Use **Method One** to solve the problem. Show all your work.

      Hint: If you can't mentally calculate what number to multiply 5 by to get 13, use a calculator or paper and pencil.

Method Two

Let's solve **number 1** *again, this time using* **Method Two**.

1. A wallet costs \$12. Sales tax of 7.5% is to be charged. To determine the amount of sales tax, a proportion can be written as follows:

$$\frac{0.075}{1.00} = \frac{x}{12.00}$$

c. The **least common multiple (LCM)** of 1.00 and 12.00 is

______________ .

Remember: The *least common multiple (LCM)* is the smallest of the common multiples of two or more numbers. For example, the LCM of 4 and 6 is 12.

d. Using the **multiplicative property of equality**, each side of the *equation* can be multiplied by this least common multiple.

Remember: The *multiplicative property of equality* states that if $a = b$, then $ac = bc$. Therefore, you can multiply (or divide) by the same number on both sides of an equation and the equation remains true.

$$\frac{0.075}{1.00} = \frac{x}{12.00}$$

$$12.00 \bullet \frac{0.075}{1.00} = 12.00 \bullet \frac{x}{12.00}$$

______________ $= x$

Let's solve **number 2** *again, this time using* **Method Two**.

2. A figure is to be enlarged. For each 2 inches in the side length of the original, there will be 5 inches in the corresponding side (or matching sides in a similar figure) of the image. If one of the side lengths in the image is 13 inches, what was the length of the corresponding side in the original?

$$\frac{2}{5} = \frac{x}{13}$$

b. Use **Method Two** to solve the problem. Show all your work.

Hint: When two numbers have no common **factor**—a number that divides into both of them with no remainder—greater than 1, the least common multiple is their *product*.

Now let's solve **number 1** *again, this time using* **Method Three**.

Method Three

1. A wallet costs $12. Sales tax of 7.5% is to be charged. To determine the amount of sales tax, a proportion can be written as follows:

$$\frac{0.075}{1.00} = \frac{x}{12.00}$$

e. You have likely used the fact that if two fractions are equal to each other, then their **cross products** are equal. To find the *cross products*,

1.00 is multiplied by ____________

and 0.075 is multiplied by ____________ .

Remember: Cross products are the **product** of one *numerator* and the opposite *denominator* in a pair of fractions.

$$\frac{0.075}{1.00} = \frac{x}{12.00}$$

1(____________) = 0.075(____________)

f. The result is that x = ____________ .

Now we'll solve **number 2** *using* **Method Three**.

2. A figure is to be enlarged. For each 2 inches in the side length of the original, there will be 5 inches in the corresponding side (or matching sides in a similar figure) of the image. If one of the side lengths in the image is 13 inches, what was the length of the corresponding side in the original?

   $\frac{2}{5} = \frac{x}{13}$

   c. Use **Method Three** to solve the problem. Show all your work.

3. The minivan driven by the writer of this problem used 5.5 gallons of gas after going 132 miles. At this rate, how much gas should be required for a distance of 330 miles?

   Your proportion could be as follows:

   $\frac{5.5}{132} = \frac{x}{330}$

   a. Use **Method One** to solve the problem. Show all your work.

      Hint: If you can't mentally calculate what number to multiply the number 132 by to get 330, use a calculator or paper and pencil.

b. Use **Method Two** to solve the problem. Show all your work.

Hint: The **prime factorization**—writing a number as the product of **prime numbers**—of 132 is 2 x 2 x 3 x 11. The *prime factorization* for 330 is 2 x 3 x 5 x 11.

132 = 2 x 2 x 3 x 11
330 = 2 x 3 x 5 x 11
LCM = 2 x 2 x 3 x 5 x 11

The LCM for 132 and 330 will be 2 x 2 x 3 x 5 x 11,

which is ____________ .)

c. Use **Method Three** to solve the problem. Show all your work.

Practice

Complete the following using the **method of your choice** *from the previous practice.*

1. A news report in August 2003 reported that 5,000 school buses traveled 363,000 miles a day to provide transportation to and from school for students in the Atlanta metro area. The average distance for each bus could be obtained by dividing 363,000 by 5,000 or by solving a proportion as follows:

   $$\frac{363{,}000}{5{,}000} = \frac{x}{1}$$

   Find the average distance for a school bus in the Atlanta metro area.

2. The news report stated that the 363,000 miles traveled each day by the school buses was far enough to circle the Earth more than 14 times.

   If this is true, the distance to circle the Earth once must be at least

   ____________ miles. Show all your work *or* explain how you solved

   the problem.

   Explanation: ____

   ____

   ____

   ____

3. The news report also stated that the 363,000 miles was enough to go to the moon and halfway back. If this is true, what is the distance to the moon? Show all your work *or* explain how you solved the problem.

Explanation: ____

____

____

____

4. The number of students in metro Atlanta public schools was reported to be more than 675,000. Of that number, 400,000 eat school lunches. What percent of students eat school lunches?

5. Metro Atlanta public schools dish up 400,000 meals each day. A chain of fast food restaurants in metro Atlanta estimate they serve 150,000 customers each day. For each 100 meals served by the restaurant chain, how many are served by public schools? A proportion could be used to solve this problem is as follows:

$$\frac{x}{100} = \frac{400,000}{150,000}$$

6. Virginia Tech's Metropolitan Institute reported a 2001-02 survey that found the ratio of drivers to vehicles in the United States to be 1.8 to 1.9. The survey reported that the typical American family has more vehicles than licensed drivers. The reporter covering the story made the following statement: "That equals 204 million vehicles and 191 million drivers." Compare these two ratios and determine whether or not the reporter's statement supported the findings. Justify your response.

   Response and justification: ________________________________

   ____

Practice

Match each definition with the correct term. Write the letter on the line provided.

______ 1. a one-dimensional measure that is the measurable property of line segments	A. decimal number
______ 2. the result of dividing two numbers	B. equation
______ 3. any number written with a decimal point in the number	C. fraction
______ 4. the constant that is multiplied by the lengths of each side of a figure that produces an image that is the same shape as the original figure	D. length (*l*)
______ 5. any part of a whole	E. percent (%)
______ 6. the comparison of two quantities	F. proportion
______ 7. a mathematical sentence stating that two ratios are equal	G. quotient
______ 8. a special-case ratio which compares numbers to 100 (the second term)	H. ratio
______ 9. any of the numbers represented by the variable	I. scale
______ 10. to find all numbers that make an equation or inequality true	J. scale factor
______ 11. a mathematical sentence in which two expressions are connected by an equality symbol	K. side
______ 12. the edge of a polygon	L. solve
______ 13. the relationship between the measures on a drawing or model and the real object	M. value (of a variable)

Practice

Use the list below to write the correct term for each definition on the line provided.

corresponding sides	**multiplicative identity**
cross product	**multiplicative property of equality**
equivalent (forms of a number)	**prime factorization**
factor	**prime number**
least common multiple (LCM)	**similar figures**

________________________ 1. the smallest of the common multiples of two or more numbers

________________________ 2. the same number expressed in different forms

________________________ 3. figures that have the same shape, have corresponding, congruent angles, and have corresponding sides that are proportional in length

________________________ 4. the product of one numerator and the opposite denominator in a pair of fractions

________________________ 5. the matching sides in similar figures

________________________ 6. if $a = b$, then $ac = bc$; if you multiply (or divide) by the same number on both sides of an equation, the equation continues to be true

________________________ 7. a number or expression that divides evenly into another number

________________________ 8. any whole number with only two whole number factors, 1 and itself

________________________ 9. writing a number as the product of prime numbers

________________________ 10. the number one (1)